AN OVERVIEW OF THE FRACTIONAL CALCULUS IN OPTICS AND QUANTUM COMPUTATION

Abstract

Fractional calculus provides a natural framework for incorporating memory, dissipation, and temporal nonlocality into quantum dynamics. In this chapter, fractional time evolution is employed to extend conventional quantum gates and entangling operations while preserving Hilbert-space structure, locality, and energetic consistency. Fractional quantum gates are shown to arise from standard local Hamiltonians acting through Mittag–Leffler propagators, leading to non-unitary yet physically meaningful operations characterized by Lévy-flight–like temporal statistics. This approach reshapes single- and two-qubit dynamics, producing gradual population transfer, algebraic dephasing, and history-dependent entanglement generation. The framework is further applied to optical and photonic systems, where fractional dynamics naturally emerge through dispersion, mode coupling, and structured propagation, offering insight into phenomena such as PSF broadening and coherence loss. Finally, implications for quantum communication and quantum key distribution are discussed, highlighting how channel memory modifies standard security assumptions and motivates memory-aware protocol design. Overall, fractional quantum dynamics are presented as a unifying theoretical lens for understanding realistic quantum operations and information processing in complex environments.

Keywords: Fractional Calculus, Fractional Quantum Dynamics, Fractional Time Evolution, Quantum Gates, Fractional Quantum Gates, Mittag–Leffler Propagators, Nonlocality, Memory Effects,

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I. INTRODUCTION

Fractional Calculus: From Mathematical Curiosity to a Language of Modern Physics

The concept of differentiation and integration of non-integer order dates back to the origins of calculus. In 1695, Gottfried Wilhelm Leibniz questioned the meaning of a "half-order derivative" in correspondence with Guillaume de l'Hôpital, anticipating both paradox and potential utility (Leibniz, 1695). Over the following centuries, this idea evolved into the formal framework now known as fractional calculus.

Key mathematical foundations were established through the works of Euler, Liouville, Riemann, Grünwald, and Letnikov, who developed consistent definitions of fractional integration and differentiation using integral transforms and limit processes (Riemann, 1847; Liouville, 1832; Grünwald, 1867; Letnikov, 1868). Despite these advances, fractional calculus remained largely peripheral to physical modeling, as integer-order differential equations were considered sufficient for most applications.

This view shifted in the late twentieth century with experimental evidence across multiple domains revealing dynamics incompatible with classical models. Viscoelastic systems exhibited long-term memory and power-law relaxation (Bagley and Torvik, 1983; Mainardi, 2010), while dielectric materials followed fractional empirical laws such as the Cole–Cole and Havriliak–Negami models (Cole and Cole, 1941; Havriliak and Negami, 1967). Similarly, anomalous diffusion processes demonstrated non-linear mean-square displacement and scale-free transport (Metzler and Klafter, 2000).

Fractional calculus provides a natural framework for such phenomena. Fractional derivatives are intrinsically nonlocal, with their values depending on the entire history of the system (Podlubny, 1999). This enables compact modeling of memory, hereditary effects, and distributed dissipation, introducing a fundamentally different description of temporal and spatial evolution.

Applications have since expanded beyond rheology and transport to include wave propagation in complex media, electromagnetic attenuation, and acoustic and seismic analysis (Caputo, 1967; Mainardi et al., 2001; Holm et al., 2013). Fractional formulations of wave and Helmholtz equations incorporate power-law attenuation and anomalous dispersion, producing non-Gaussian responses and scale-dependent propagation (Szabo, 1994; Laskin, 2002). These effects are particularly relevant in optical and astrophysical contexts involving turbulent or fractal media.

Fractional approaches have also influenced quantum physics, especially in systems exhibiting non-Markovian dynamics with memory and information backflow (Breuer and Petruccione, 2002; Rivas et al., 2014). Fractional time-evolution equations provide effective descriptions of such behavior and are increasingly explored in quantum optics and quantum information science (Laskin, 2000; Naber, 2004).

This chapter adopts the view that fractional calculus serves as a unifying framework for describing energy dissipation, wave propagation, and information transport in systems governed by memory and scale invariance. Emphasis is placed on physical intuition and cross-disciplinary connections, highlighting its role as a conceptual tool in contemporary physics.

II. FRACTIONAL DERIVATIVES AND INTEGRALS: MEMORY AND DISSIPATION

1. From Local to Nonlocal Calculus

In classical physics, time evolution is typically described using integer-order derivatives. A first-order derivative represents an instantaneous rate of change, while higher-order derivatives describe local curvature or acceleration. A key implicit assumption behind such formulations is **locality in time**: the future evolution of a system depends only on its present state (and possibly a finite number of its immediate derivatives).

Fractional calculus relaxes this assumption. Fractional derivatives and integrals extend the concept of differentiation and integration to non-integer orders, but in doing so they introduce a fundamental change in the nature of temporal and spatial operators. Unlike integer-order derivatives, fractional operators are inherently **nonlocal**, meaning they incorporate information from an extended interval of time or space rather than an infinitesimal neighborhood.

This nonlocality is not an abstract mathematical artifact; it has direct physical consequences. In particular, it provides a natural way to describe **memory effects** and **distributed dissipation**, which are ubiquitous in complex and multiscale systems.

Nonlocality as a Change in the Dimensional Nature of Variation

In classical calculus, the derivative $(t)/dt$ quantifies the change of a function with respect to an infinitesimal change in time. Although the interval dt is taken to be vanishingly small, the operation remains strictly local and one-dimensional in nature: the derivative probes variation along the time axis at a single instant. The infinitesimal limit does not alter the dimensional character of the evolution; it merely refines resolution within the same temporal dimension. Fractional calculus introduces a fundamentally different concept of variation. A fractional derivative of order $0 < \alpha < 1$ cannot be interpreted as a derivative over a smaller time interval, but rather as an operation that integrates contributions over an extended temporal domain with a scale-free weighting. In this sense, fractional differentiation effectively interpolates between dimensions of evolution, replacing pointwise change with a continuum of temporal influences. The resulting dynamics no longer reside purely on the time axis but instead unfold over a history-extended space, whose effective dimensionality reflects the order of the fractional operator. Nonlocality in fractional calculus therefore corresponds not merely to memory in time, but to a deeper modification of how change itself is defined—one in which evolution acquires a fractional, history-spanning dimensional character rather than remaining confined to an infinitesimal neighborhood.

2. Fractional Integrals as Weighted Memory Operators

A convenient starting point is the fractional integral. The Riemann–Liouville fractional integral of order α >0 is defined as

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int (t-\tau) \qquad f(\tau) d\tau$$

This expression makes the interpretation immediately clear. The value of the fractional integral at time t is a **weighted accumulation of the entire past history** of the function (τ), with a kernel proportional to $(t-\tau)^{\alpha-1}$. Unlike ordinary integration, which treats all past contributions uniformly, the fractional integral assigns **scale-free weights** that decay algebraically with time.

Such power-law weighting implies that past events are never fully forgotten. Even contributions from the distant past continue to influence the present, albeit with diminishing strength. This behavior contrasts sharply with exponential memory kernels, which introduce a characteristic relaxation time beyond which past information becomes negligible.

In physical terms, fractional integrals naturally model systems with **long-term memory**, such as viscoelastic materials, charge transport in disordered media, and wave propagation in fractal environments.

3. Fractional Derivatives: History-Dependent Rates of Change

Fractional derivatives can be constructed by combining fractional integration with ordinary differentiation. Among several equivalent definitions, the Caputo fractional derivative is particularly useful for physical applications, as it allows for conventional initial conditions. For $0 < \alpha < 1$, it is defined as

$$ {}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{df(\tau)}{d\tau} \frac{d\tau}{(t-\tau)^{\alpha}} $$

This expression reveals a crucial property: a fractional derivative is not a pointwise operation. Instead, it represents a **history-weighted average of the ordinary derivative** over all past times. The kernel $(t-\tau)^{-\alpha}$ decays as a power law, ensuring that the influence of past states persists over long durations.

From a physical perspective, the fractional derivative measures a rate of change that is continuously influenced by previous evolution. The present response of the system is therefore shaped not only by its current state but also by how it arrived there.

4. Memory Kernels and the Origin of Long-Time Correlations

The integral form of fractional derivatives highlights their connection to memory kernels. In general, a dynamical equation with memory can be written as

$$ \frac{d}{dt} x(t) = \int_0^t K(t-\tau) F[x(\tau)] d\tau $$

When the memory kernel has an exponential form, $K(t) \sim e^{-t/\tau}$, the system exhibits short-term memory with a well-defined relaxation time. In contrast, fractional derivatives correspond to kernels of the form

$$ (t) \sim t^{-\alpha}, 0 < \alpha < 1, $$

which decay slowly and lack any intrinsic time scale. This scale-free behavior is a hallmark of systems with hierarchical or fractal internal structure.

As a result, fractional models naturally capture **long-time correlations**, aging effects, and slow relaxation processes that are inaccessible to integer-order descriptions.

Consider a standard first-order relaxation equation:

$$\frac{dx}{dt} = -\lambda t$$

It gives the typical solution

$$x(t) = x(0)e^{-\lambda t}$$

The system

- Forgets its past exponentially fast
- Has a **single time scale** $1/\lambda$
- Is **Markovian**

Now, replace the first-order derivative with a fractional one:

$${}^{C}D_t^{\alpha}x(t) = -\lambda x(t),\, 0 < \alpha < 1$$

Its solution can be expressed as

$$\mathrm{x}(t) = x(0)E_{\alpha}(-\lambda t^{\alpha})$$

where, $E_{\alpha}(.)$ is **Mittag–Leffler function**.

Asymptotically:

$$E_{\alpha}(-\lambda t^{\alpha}) \sim \frac{1}{\lambda t^{\alpha}},\, t \to \infty$$

Physical Meaning

- Decay is **algebraic**, not exponential
- Past states influence the present for very long times
- The system has **memory with no characteristic time scale**

5. Fractional Dissipation: Beyond Instantaneous Friction

The connection between fractional calculus and dissipation becomes particularly transparent when examining damping forces. In classical mechanics, viscous damping is modeled as

$$F_{diss}(t) = -\gamma \dot{x}(t)$$

implying that energy loss at a given instant depends only on the instantaneous velocity. This assumption fails in many real materials, where dissipation depends on the past deformation history.

Replacing the velocity with a fractional derivative leads to

$$F_{diss}(t) = -\gamma\, {}^{C}D_t^{\alpha} x(t)\,, \quad 0 < \alpha < 1.$$

In this formulation, the dissipative force depends on the entire velocity history of the system. Energy is not lost instantaneously but is instead **distributed over time**, reflecting delayed internal relaxation mechanisms. Such fractional damping models successfully describe experimentally observed power-law energy loss in viscoelastic solids, polymers, biological tissues, and complex fluids.

Importantly, fractional dissipation does not imply stronger damping in the conventional sense. Rather, it introduces a qualitatively different form of energy loss, characterized by slow decay and persistent memory.

Energy Dissipation: Why Fractional Systems Lose Energy Differently

Classical Damping

For a damped oscillator:

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

The corresponding energy dissipation rate of this system:

$$dE/dt = -\gamma\dot{x}^2$$

This shows dissipation is **instantaneous** and proportional to velocity.

Fractional Damping

Now replace the classical damping term with fractional damping:

$$m\ddot{x} + \gamma\, {}^{C}D_t^{\alpha} x + kx = 0,\, 0 < \alpha < 1$$

where the damping force is:

$$F_d(t) = \gamma\, {}^{C}D_t^{\alpha} x(t)$$

Substitute the definition:

$$F_d(t) = \frac{\gamma}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}\, d\tau}{(t-\tau)^{\alpha}}$$

Interpretation
- The damping force depends on **past velocities**
- Energy dissipation is **history-dependent**
- The system “remembers” how it moved before losing energy

Fractional damping = distributed friction in time

6. Frequency-Domain Perspective: Storage And Loss Intertwineda

The physical implications of fractional derivatives become even clearer in the frequency domain. Taking the Fourier transform of a fractional derivative yields

$$\mathcal{F}\left[\,{}^{C}D_t^{\alpha} f(t)\right] = (i\omega)^{\alpha} F(\omega)$$

The complex power $(i\omega)$ can be written as

$$(i\omega)^{\alpha} = \omega^{\alpha}\left\{\cos\left(\frac{\pi\alpha}{2}\right) + i\sin\left(\frac{\pi\alpha}{2}\right)\right\}$$

This decomposition reveals that fractional operators inherently couple **energy storage** (real part) and **energy dissipation** (imaginary part). Unlike integer-order models, where stiffness and damping appear as separate terms, fractional models blend these effects in a frequency-dependent manner.

This intrinsic coupling explains why fractional calculus provides remarkably accurate descriptions of experimentally measured loss tangents (i.e. loss/storage $\sim\omega^{\alpha^{-1}}$) and quality factors over broad frequency ranges.

7. Implications for Wave Phenomena and Imaging

When fractional derivatives are incorporated into wave equations, dissipation and dispersion become inseparable. Power-law attenuation, anomalous phase delay, and non-Gaussian impulse responses emerge naturally. These effects have direct consequences for imaging systems, where wave coherence and phase stability determine resolution and contrast.

In particular, the presence of long-term memory alters the effective point spread function of an imaging system, leading to scale-dependent broadening and heavy-tailed sidelobes. As will be discussed in later sections, such fractional propagation effects interact nontrivially with super-oscillatory fields and sparse-aperture imaging architectures, especially in optical and astrophysical contexts.

Remark

Fractional derivatives and integrals provide more than a mathematical generalization of classical calculus. They encode a fundamentally different physical picture in which evolution is shaped by memory, dissipation is distributed over time, and scale invariance replaces characteristic relaxation scales. This perspective forms the foundation for understanding fractional wave propagation, anomalous imaging behavior, and memory-driven dynamics in both classical and quantum systems.

III. FRACTIONAL WAVE EQUATIONS: PROPAGATION WITH MEMORY

Wave equations lie at the heart of physics, governing phenomena ranging from acoustics and optics to electromagnetic and quantum waves. In their classical form, wave equations assume that propagation is local and either lossless or subject to instantaneous damping. Such assumptions are often violated when waves propagate through complex, heterogeneous, or multiscale media (Caputo, 1967; Mainardi, 2010).

A classical wave equation may be written as

$$\frac{\partial^2 u(\boldsymbol{r},t)}{\partial t^2} = c^2\nabla^2 u(\boldsymbol{r},t)$$

which is time-reversal invariant and conserves energy. Any attempt to incorporate memory or anomalous dissipation must therefore modify the operator structure of this equation.

Fractional wave equations accomplish this by replacing or augmenting integer-order derivatives with fractional ones. A representative time-fractional wave equation is

$$\frac{\partial^2 u}{\partial t^2} + \gamma D_t^{\alpha} u = c^2\nabla^2 u, \qquad 0 < \alpha < 1$$

where the fractional derivative introduces temporal nonlocality (Caputo, 1967; Szabo, 1994; Holm et al., 2013). In this formulation, the wave field at a given time depends on its entire prior evolution, reflecting a medium with memory and distributed relaxation times.

Such equations naturally give rise to power-law attenuation and anomalous dispersion, phenomena widely observed in acoustics, electromagnetism, and geophysical wave propagation (Mainardi et al., 2001; Holm, 2019).

1. Fractional Point Spread Function Broadening

The influence of fractional wave propagation becomes particularly evident in imaging systems, where the point spread function (PSF) encodes the system’s response to a point source and determines resolution limits (Goodman, 2005).

In conventional diffraction theory, the PSF is determined by the aperture and wavelength and typically exhibits rapidly decaying sidelobes. Even when turbulence or aberrations are present, the resulting PSFs are often modeled using Gaussian or exponentially decaying statistics (Roddier, 1981).

Fractional propagation alters this behavior fundamentally. Because fractional operators introduce long-range temporal or spatial correlations, the impulse response of the system acquires **heavy-tailed** characteristics. The effective PSF can be viewed as a convolution,

$$PSF_{eff} = PSF_{ideal} * K_{\alpha}$$

where K_{α} is a scale-free kernel associated with the fractional operator (Mainardi, 2010; Metzler and Klafter, 2000). This convolution redistributes energy across spatial scales, producing algebraic sidelobes rather than exponentially decaying ones.

As a result, resolution becomes inherently scale-dependent. While the central lobe may remain narrow, extended sidelobes can dominate the energy distribution, an effect that has important implications for high-resolution and super-oscillatory imaging systems (Berry and Popescu, 2006; Rogers et al., 2012).

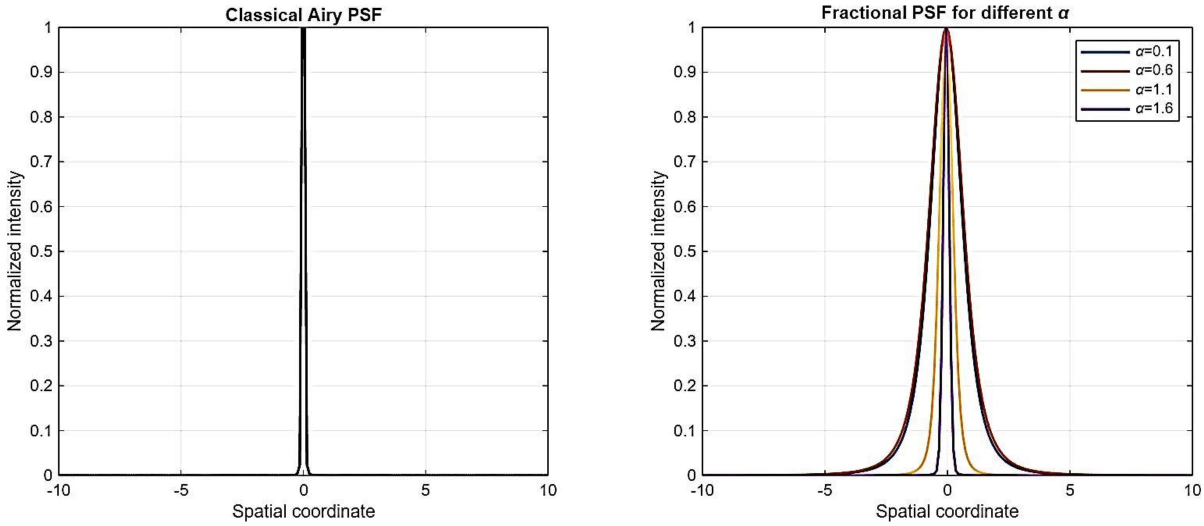

Figure 1: Comparison between a Classical Airy Point Spread Function and a Point Spread Function Generated Directly from a Fractional Wave Operator. While the Classical PSF Exhibits Well-Defined Diffraction Rings and Rapid Sidelobe Decay, the Fractional PSF Displays Heavy-Tailed Behaviour (Much Slower Decay Than Airy Sidelobes), Reflecting the Nonlocal and Memory-Driven Nature Of Fractional Propagation. the Absence of a Characteristic Decay Length Highlights the Emergence of Scale-Dependent Resolution.

2. Energy Dissipation in Fractional Wave Propagation

The PSF broadening described above originates from the underlying mechanism of **fractional dissipation**. In classical wave theory, dissipation is usually introduced through local damping terms, leading to exponential attenuation and well-defined absorption coefficients.

Fractional wave equations describe a fundamentally different form of energy loss. The inclusion of a fractional derivative implies that dissipation depends on the wave's past evolution, corresponding to a medium with internal relaxation processes spanning a broad range of time scales (Bagley and Torvik, 1983; Caputo, 1967).

In the frequency domain, this leads to power-law attenuation of the form

$$\alpha(\omega) \propto \omega^{\alpha}$$

a behavior observed experimentally in a wide variety of lossy and dispersive media, including biological tissue, polymers, and geological materials (Szabo, 1994; Holm et al., 2013). Importantly, fractional dissipation intrinsically couples attenuation and dispersion, making it impossible to treat energy loss as a purely local phenomenon.

This coupling has direct consequences for wave coherence and phase stability, which are critical parameters in imaging and interferometric systems.

3. Optical and Astrophysical Imaging Implications

Fractional wave dynamics are particularly relevant in optical and astrophysical imaging, where waves propagate through media characterized by turbulence, plasma fluctuations, and multiscale inhomogeneities. Atmospheric turbulence, interstellar plasma, and gravitational perturbations all introduce long-range correlations that challenge classical propagation models (Tatarskii, 1961; Narayan, 1992).

Observationally, these effects manifest as:
- Anomalous PSF broadening,
- Non-Gaussian scattering halos,
- Long-lived phase distortions.

Fractional propagation models provide a unified framework for describing such phenomena without introducing ad hoc corrections (Mainardi, 2010; Metzler et al., 2014). In astronomical imaging systems such as sparse-aperture interferometers and hypertelescopes, these effects place practical limits on resolution that go beyond classical diffraction theory.

Super-oscillatory imaging further accentuates the importance of fractional effects. Because super-oscillatory fields rely on delicate interference among band-limited components, they are especially sensitive to long-memory propagation. Fractional PSF broadening therefore plays a decisive role in determining the stability and contrast of super-oscillatory features in realistic optical and astrophysical environments (Berry, 2012; Dennis et al., 2014).

4. Super-Oscillation in the Presence of Fractional Wave Propagation

Super-oscillatory fields are characterized by local oscillations that exceed the highest spatial frequency present in their Fourier spectrum, achieved through precise interference among band-limited components. In classical wave propagation, such structures are accompanied by extended sidelobes but remain predictable within the framework of integer-order diffraction theory. When propagation is governed by a fractional wave equation, however, the super-oscillatory phenomenon acquires a fundamentally new character. The nonlocal nature of fractional operators modifies the Green's function of the system, leading to point spread functions with heavy-tailed profiles and scale-free energy redistribution (i.e. not governed by a single characteristic scale, rather it is redistributed over spatial scales). As a consequence, super-oscillatory cores may persist locally—often retaining sub-diffraction features—while the surrounding energy is redistributed over extended spatial scales. This decoupling between local oscillatory behaviour and global energy confinement highlights a critical distinction: fractional propagation does not suppress super-oscillation, but alters its stability, contrast, and energetic cost. In such systems, resolution becomes an emergent, scale-dependent property shaped by memory and nonlocality, rather than a fixed limit imposed solely by aperture geometry or bandwidth.

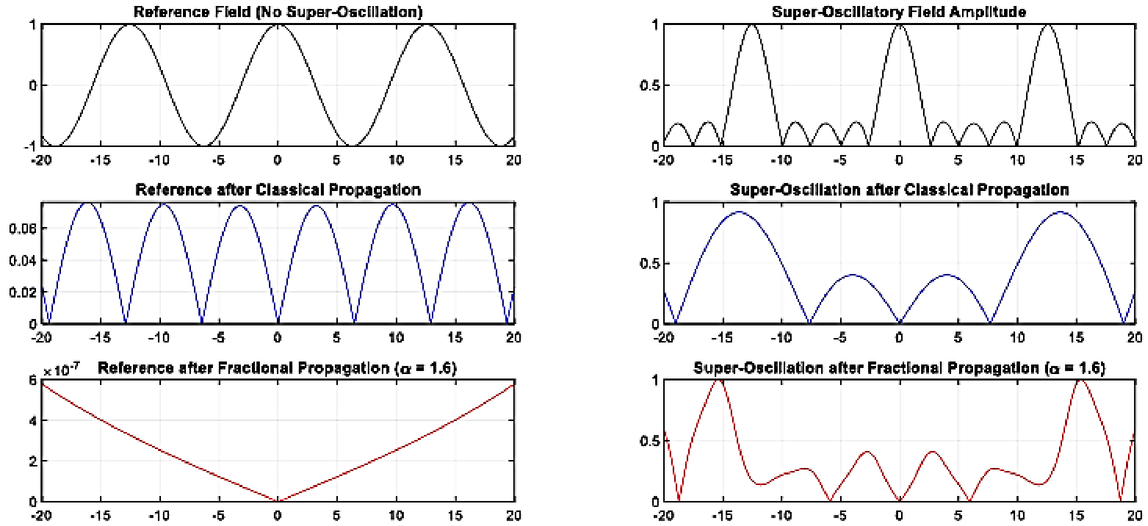

Figure 2: The Inclusion 0f A Non–Super-Oscillatory Reference Field Demonstrates that Heavy-Tailed Broadening Arises Intrinsically from Fractional Propagation, While Super-Oscillation Selectively Amplifies the Sensitivity of Local Features to this Scale-Free Energy Redistribution.

IV. FRACTIONAL CALCULUS IN QUANTUM MECHANICS

1. Motivation: Why Fractional Quantum Mechanics?

Conventional quantum mechanics is formulated using local differential operators and assumes Markovian time evolution. While this framework has been extraordinarily successful for isolated systems, it becomes increasingly limited when quantum dynamics are influenced by complex environments, long-range interactions, or structured continua. In such situations, memory effects, anomalous transport, and nonlocal correlations cannot be consistently captured by integer-order operators alone (Caldeira & Leggett, 1983; Breuer & Petruccione, 2002).

Fractional calculus provides a natural extension of quantum theory by embedding nonlocality directly into the fundamental operators governing evolution. Rather than introducing dissipation or decoherence phenomenologically, fractional formulations encode these effects at the level of the Schrödinger equation itself, enabling a unified description of memory and long-range quantum dynamics (Laskin, 2000; Herrmann, 2014).

2. Classical Schrödinger Equation: A Local Theory

The standard time-dependent Schrödinger equation is,

$$ih\frac{\partial\psi(r,t)}{\partial t} = \left[-\frac{h^2}{2m}\nabla^2 + V(r,t)\right]\Psi(r,t)$$

where, $\psi(\boldsymbol{r},\, t)$ is the wavefunction describing the probability amplitude of finding a particle with rest mass m at position $\boldsymbol{r}$ and time t. $\hbar$ is the reduced Plank's constant. $(\boldsymbol{r},\, t)$ is the potential energy of the system. The equation can be expressed in a more compact form with operator representat~~ion~~ $E|\psi\rangle = \widehat{H}|\psi\rangle$ where $\widehat{H} = \left[-\ \frac{\hbar^2}{2m}\nabla^2 + v(r,t)\right]\psi(r,t)$ is the Hamiltonian operator representing the total kinetic and potential energy. E is the energy operator.

This equation is inherently local in both time and space. The Laplacian operator (∇^2) couples only infinitesimally neighboring points, leading to quadratic dispersion relations and well-defined localization scales (Sakurai & Napolitano, 2017).

This formulation implies:

- Strictly local spatial propagation,
- Unitary time evolution,
- Absence of intrinsic memory,
- Energy spectra governed by $E \propto k^2$, where k is the wavevector. More precisely $E = \frac{\hbar^2 k^2}{2m}$

Such assumptions are violated in systems exhibiting anomalous diffusion, long-range hopping, or strong environmental coupling (Metzler & Klafter, 2000; Zaslavsky, 2002).

3. Fractional Schrödinger Equation: Core Idea

A major step toward fractional quantum mechanics was introduced by Laskin, who derived a **space-fractional Schrödinger equation** by generalizing the Feynman path integral to Lévy flight trajectories (Laskin, 2000; Laskin, 2002):

$$i\hbar\frac{\partial\psi}{\partial t} = D_\alpha(-\hbar^2\nabla^2)^{\frac{\alpha}{2}}\psi + V(\boldsymbol{r})\psi$$

Here α is the Levy index having a range (1,2]. D_α is a scale constant having dimension $[D_\alpha] = [energy]^{1-\alpha}[length]^\alpha[time]^\alpha$. For α = 2, we retrieve the standard Schrodinger equation form, i.e. $D_2 = 1/2m$, where m is particle's mass. In spectrum plane, i.e. in terms of k-space this relation can be expressed as

$$i\partial_t\psi(k,\ t)\ =\ D_\alpha|k|^\alpha\psi(k,\ t)$$

Physically, this implies that quantum amplitudes propagate through long-range spatial coupling, resembling Lévy-type quantum transport rather than Brownian motion (Laskin, 2000; Herrmann, 2014).

4. Fractional Operators and Physical Interpretation Fractional Laplacian

The fractional Laplacian is most transparently defined in Fourier space:

$$\mathcal{F}\{(-\nabla^2)^{\alpha/2}\psi\} = |k|^\alpha\mathcal{F}\{\psi\}$$

This definition highlights several key properties

- Non-quadratic dispersion relations,
- Scale-free weighting of spatial frequencies,
- Absence of a unique localization length.

Such operators naturally produce heavy-tailed wavefunctions and anomalous energy distributions, closely paralleling the fractional wave propagation discussed earlier in optical systems (Metzler & Klafter, 2000; Mainardi, 2010).

Time-Fractional Schrödinger Equation

An alternative generalization involves replacing the first-order time derivative with a fractional derivative:

$$(i\hbar)^{\beta} D_t^{\beta} \psi = \widehat{H}\psi, \;\; 0 < \beta \leq 1$$

Where D_t^{β} is a fractional time derivative.

Time-fractional Schrödinger equations introduce intrinsic memory and generally lead to non-unitary evolution, reflecting effective dynamics of open quantum systems rather than closed ones (Naber, 2004; Laskin, 2018). Such formulations have been applied to quantum transport, dissipative dynamics, and decoherence processes.

Energy, Probability, and Interpretation

A critical distinction must be made between space- and time-fractional quantum models. Space-fractional Schrödinger equations can preserve probability conservation and Hermiticity under appropriate conditions, whereas time-fractional equations typically describe reduced dynamics with probability leakage into unobserved degrees of freedom (Breuer & Petruccione, 2002; Herrmann, 2014).

This mirrors the role of fractional operators in classical wave physics, where dissipation and memory emerge naturally from nonlocal temporal kernels rather than explicit damping terms (Caputo, 1967; Mainardi, 2010).

Scale-Free Energy and Super-Oscillation in Quantum Systems

Fractional quantum dynamics lead to wavefunctions whose energy and probability densities lack a single characteristic scale. This scale-free redistribution is a direct consequence of nonlocal operators and power-law kernels, producing heavy-tailed spatial profiles and anomalous dispersion (Zaslavsky, 2002).

Super-oscillatory quantum states, though formally band-limited, become particularly sensitive to such dynamics. While local rapid oscillations may persist, their energetic cost is redistributed across extended spatial and momentum scales, reducing contrast and stability in fractional quantum environments. This behavior reflects a deep connection between super-oscillation, nonlocal propagation, and quantum memory effects (Berry & Popescu, 2006; Berry, 2012).

5. Lévy Flights, Brownian Motion, and the Origin of Fractional Quantum Dynamics

a. Brownian Motion: The Classical Quantum Path Picture

In standard quantum mechanics, the Feynman path integral formulation provides a powerful interpretation of wavefunction evolution. The propagator is obtained by summing over all possible paths, each weighted by the classical action. Crucially, these paths are assumed to be

continuous and locally constrained, resembling Brownian motion (Feynman & Hibbs, 1965).

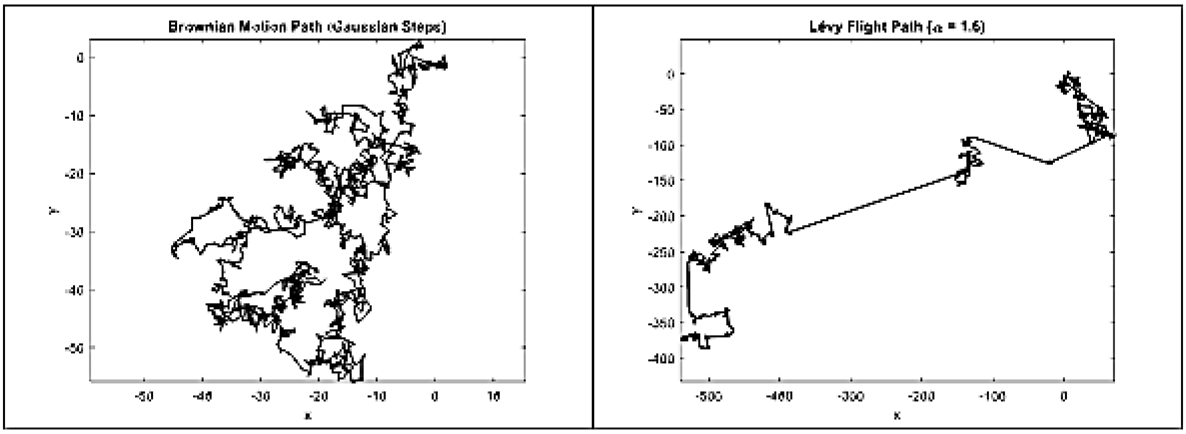

Figure 3: Spatial Trajectories Illustrating Two Fundamentally Different Stochastic Transport Mechanisms. (A) Brownian Motion Path Generated Using Gaussian-Distributed Step Lengths, Characterized By Continuous, Locally Constrained Exploration of Space. (B) Lévy Flight Path Generated Using a Power-Law Step-Length Distribution, Dominated by Rare but Long-Range Jumps and Exhibiting Scale-Free Transport. Brownian Paths Underpin the Standard Schrödinger Equation with a Local Laplacian Operator, While Lévy Paths Give Rise to the Fractional Schrödinger Equation Through the Emergence of Nonlocal Kinetic Operators.

Brownian motion is characterized by:

- Short, incremental steps,
- Finite variance of step lengths,
- Gaussian probability distributions,
- A Well-defined diffusion scale.

These properties lead naturally to the **Laplacian operator** in the Schrödinger equation and to the familiar quadratic dispersion relation $E \propto k^2$. As a result, standard quantum mechanics describes probability transport that is local, scale-dependent, and dominated by nearby spatial contributions.

b. Lévy Flights: A Fundamentally Different Transport Mechanism

Lévy flights generalize Brownian motion by allowing **rare but arbitrarily long jumps**. Instead of Gaussian step-length distributions, Lévy processes are governed by **power-law probability distributions** with divergent second moments (Metzler & Klafter, 2000).

Key characteristics of Lévy flights include:

- Heavy-tailed jump-length distributions,
- Absence of a characteristic step size,
- Scale-free transport,
- Dominance of rare, long-range events.

Physically, Lévy flights describe motion in which long-distance transitions are not suppressed exponentially but occur with algebraically decreasing probability. Such behavior is widely observed in turbulent transport, plasma physics, chaotic systems, and complex media (Zaslavsky, 2002).

A more intuitive distinction between Brownian motion and Lévy flights emerges when the stochastic trajectories are represented as spatial zigzag paths rather than cumulative displacements. Brownian motion explores space through a dense succession of small, locally constrained steps, producing a continuous and highly folded trajectory. In contrast, Lévy flights exhibit predominantly local motion interrupted by rare but extremely long jumps, resulting in trajectories dominated by discontinuous spatial excursions. This qualitative difference reflects the underlying statistics of the motion and directly maps onto the distinction between the local Laplacian operator in the conventional Schrödinger equation and the nonlocal fractional Laplacian appearing in fractional quantum mechanics.

Occasional long jumps introduce a power-law tail in the displacement distribution, eliminating any characteristic length scale and invalidating the central limit theorem. As a result, the effective kinetic operator governing the motion becomes nonlocal and is naturally expressed as a fractional power of the Laplacian.

c. From Lévy Flights to the Fractional Schrödinger Equation

A decisive connection between Lévy flights and quantum mechanics was established by Laskin, who reformulated the Feynman path integral by replacing Brownian paths with Lévy trajectories (Laskin, 2000; 2002). This modification leads directly to the **fractional Schrödinger equation**:

$$i\hbar \frac{\partial \psi}{\partial t} = D_\alpha (-\hbar^2 \nabla^2)^{\frac{\alpha}{2}} \psi + V(\boldsymbol{r})\psi$$

Here:

- α = 2 corresponds to Brownian motion and recovers standard quantum mechanics,
- α < 2 corresponds to Lévy flights and introduces nonlocal spatial coupling.

Thus, the fractional Laplacian is not an abstract mathematical replacement, but a **direct statistical signature of Lévy-type quantum paths**.

Brownian vs Lévy Quantum Transport: A Direct Comparison

Feature	Brownian Motion	Lévy Flight
Step length distribution	Gaussian	Power-law
Variance	Finite	Divergent
Characteristic scale	Present	Absent
Quantum operator	Laplacian (∇^2)	Fractional Laplacian $(-\nabla^2)^{\alpha/2}$
Schrödinger equation	Standard	Fractional
Energy transport	Local	Nonlocal
Wavefunction tails	Exponential / Gaussian	Heavy-tailed

This comparison clarifies why **fractional quantum mechanics naturally produces scale-free energy distributions**, just as you observed earlier in fractional wave propagation and PSF broadening.

d. Physical Interpretation and Quantum Meaning

In the fractional Schrödinger framework, a quantum particle does not explore space through smooth, incremental motion. Instead, its probability amplitude spreads through **sporadic, long-range transitions**, reflecting a fundamentally different mode of quantum exploration.

This interpretation does **not** imply superluminal motion or violation of causality. Rather, it reflects an **effective description** of quantum dynamics in structured or continuum-coupled environments, where intermediate degrees of freedom have been integrated out.

In this sense:

- Brownian quantum motion relates to isolated, weakly interacting systems
- Lévy quantum motion indicates complex, multiscale, or non-Markovian systems

6. Time-Smearing as Fractional Hamiltonian Action (not a new Hamiltonian)

In fractional quantum dynamics, the Hamiltonian itself remains a well-defined, time-independent operator corresponding to the system's physical energy. What changes is how its action accumulates in time. The fractional Schrödinger equation can be rewritten formally as a **convolution in time**:

$$\mid \Psi(t)\rangle = \int_0^t K_\alpha(t-\tau)\, e^{-iH(t-\tau)/\hbar} \mid \Psi(\tau)\rangle d\tau,$$

where(t) is a memory kernel with power-law tails, reflecting Lévy waiting-time statistics. This expression makes explicit that the system does not respond instantaneously to the Hamiltonian; instead, the effect of the Hamiltonian is **smeared over its past history**.

It is tempting to interpret this as a time-dependent Hamiltonian. However, this time dependence does **not** correspond to an externally driven or fluctuating energy operator. Rather, it encodes how the system *samples* its own Hamiltonian over time.

a. Why Time Smearing does not Violate Energy Conservation

Energy conservation in quantum mechanics is tied to the **Hermiticity of the Hamiltonian** and the absence of explicit time dependence in the energy operator itself. In fractional dynamics:

- The Hamiltonian H remains Hermitian,
- No new terms are added to its spectrum,
- No external work is performed on the system.

The apparent dissipation arises because the evolution is **nonlocal in time**, not because energy is lost to an environment. Fractional time derivatives effectively distribute energy exchange over long timescales, producing algebraic decay of coherent oscillations. However, this redistribution does not change the expectation value of the Hamiltonian in the full, memory-aware description of the system.

In other words, energy is conserved **globally in time**, even though it may not be conserved **instantaneously** in the usual Markovian sense.

The time-smearing inherent in fractional quantum dynamics reflects a memory-weighted accumulation of the Hamiltonian's action rather than a modification of the Hamiltonian itself. Since the Hamiltonian remains Hermitian and time independent, the underlying energy spectrum of the system is preserved. Apparent dissipation arises from the nonlocality of the temporal evolution, which redistributes energy exchange over long timescales in a Lévy-flight–like manner, rather than from true energy loss or coupling to additional degrees of freedom. Energy conservation therefore holds globally, even though instantaneous dynamics deviate from the Markovian picture.

b. Connection to Super-Oscillation and Scale-Free Energy

The Lévy-flight interpretation provides a natural explanation for the fragility of super-oscillatory structures under fractional propagation. While super-oscillations rely on delicate local interference, Lévy-type transport redistributes probability amplitude across extended spatial scales, diluting contrast and stability.

Thus, fractional quantum mechanics does not eliminate super-oscillations but alters the **energetic bookkeeping** underlying their formation, mirroring the behavior observed earlier in fractional optical PSFs.

Conceptual Bridge to Quantum Continuum and Computation

Because Lévy-type quantum dynamics emerge from nonlocal transport and memory effects, they offer a powerful framework for describing quantum continua, anomalous transport in networks, and non-Markovian information flow. These features are increasingly relevant for quantum technologies operating beyond idealized, closed-system assumptions.

7. Quantum Continuum and Fractional Dynamics

a. What Is Meant by a Quantum Continuum?

In standard quantum mechanics, systems are often idealized as either discrete (finite-dimensional Hilbert spaces, isolated bound states) or weakly perturbed by their surroundings. In reality, many physically relevant systems interact with a **continuum of degrees of freedom**, such as phonon baths, electromagnetic modes, disordered lattices, or extended reservoirs.

A *quantum continuum* refers to a situation where the system is not coupled to a small, well-separated set of modes, but rather to a dense or effectively infinite spectrum of states. In such cases, energy, probability amplitude, and information can flow across a wide range of scales, and the system's dynamics can no longer be described by purely local or memoryless evolution.

b. Breakdown of Markovian Dynamics

When a quantum system is coupled to a continuum, the assumption of Markovian evolution — that the future depends only on the present state — generally breaks down. Instead, the system retains a **memory of its past interactions**, mediated through the continuum.

Mathematically, this manifests as:

- Temporal convolution kernels,
- Non-exponential decay laws,

- Long-time correlations,
- Deviation from simple Lindblad-type dynamics.

Fractional calculus provides a compact and physically meaningful way to encode such memory effects through nonlocal operators, replacing exponential relaxation with power-law behavior.

c. Fractional Time Evolution and Quantum Memory

In continuum-coupled systems, the effective time evolution of the reduced quantum state often exhibits **power-law memory kernels**, rather than delta-correlated noise. This behavior is naturally captured by time-fractional derivatives:

$$D_t^{\beta}\psi, \quad 0 < \beta < 1$$

which weight past states with algebraically decaying influence. Physically, this means that:

- The system never fully “forgets” its earlier evolution,
- Decoherence and dissipation occur gradually across scales,
- Relaxation times are not uniquely defined.

This mirrors the role of fractional temporal operators in classical dissipative systems and fractional wave equations discussed earlier in the chapter.

d. Spatial Continuum and Nonlocal Transport

Beyond temporal memory, coupling to a spatial continuum also alters how quantum amplitudes propagate. In structured or disordered continua, probability transport may occur through long-range hopping, resonant tunneling, or collective modes, rather than through smooth local diffusion.

Such transport mechanisms naturally lead to:

- Lévy-type spatial statistics,
- Heavy-tailed wavefunctions,
- Fractional Laplacian operators in the effective Hamiltonian.

Thus, the fractional Schrödinger equation emerges as an effective description of quantum motion within a continuum, rather than as a fundamental modification of quantum mechanics itself.

e. Energy Flow, Scale-Free Spectra, and Continuum Coupling

A defining feature of quantum continua is the absence of a single dominant energy scale. Energy injected into the system can spread across the continuum in a **scale-free manner**, producing power-law spectra and anomalous relaxation dynamics.

This behavior closely parallels:

- Scale-free PSF broadening in fractional optics,
- Energy redistribution in super-oscillatory fields,
- Heavy-tailed probability densities in Lévy flights.

In all these cases, fractional operators arise because the system's response spans multiple scales simultaneously.

f. Conceptual Link to Measurement and Decoherence

Quantum continua play a central role in decoherence and measurement theory. When a quantum system interacts with a continuum, phase information becomes distributed across many degrees of freedom, leading to effective classicality.

Fractional dynamics offer a way to describe this process without assuming instantaneous decoherence or purely exponential decay. Instead, coherence loss becomes a **gradual, scale-dependent process**, consistent with experimental observations in complex quantum environments.

g. Why the Quantum Continuum Matters

The concept of a quantum continuum is not merely of theoretical interest. It is directly relevant to:

- Solid-state quantum devices,
- Photonic and plasmonic systems,
- Cold atoms in optical lattices,
- Quantum transport in disordered media,
- And emerging quantum technologies operating far from ideal isolation.

In such systems, fractional dynamics provide a bridge between microscopic quantum laws and emergent macroscopic behavior.

Fractional calculus thus offers a natural language for describing quantum systems embedded in continua, where memory, nonlocality, and scale-free energy transport fundamentally shape the dynamics.

8. Fractional Decoherence and Coherence Protection

In quantum systems coupled to a continuum, decoherence is traditionally modelled as an exponential loss of phase coherence resulting from memoryless environmental interactions. Fractional dynamics modify this picture by introducing long-time memory and nonlocal temporal correlations, leading to **non-exponential, power-law decoherence**. In such systems, coherence is not destroyed abruptly but is gradually redistributed across extended time scales, allowing partial phase information to persist for longer durations. This fractional decoherence mechanism can, under certain conditions, slow the effective loss of coherence and enhance robustness against environmental noise, not by eliminating dissipation but by spreading it across scales. Consequently, coherence protection in fractional quantum systems arises as an emergent effect of memory and nonlocality, rather than from isolation or error suppression, offering a fundamentally different perspective on maintaining quantum coherence in complex environments.

a. Fractional Dynamics and Entanglement in Quantum Continua

Entanglement is intrinsically sensitive to environmental coupling, as correlations between subsystems can be rapidly degraded when quantum information leaks into a surrounding continuum. In conventional Markovian environments, this degradation typically occurs exponentially, leading to phenomena such as entanglement sudden death. When the system–environment interaction exhibits long-range temporal or spatial correlations, however, entanglement dynamics deviate markedly from this simple picture. Fractional time evolution, arising from memory-bearing continua, naturally leads to **non-exponential decay and revival-like behavior of entanglement**, reflecting the persistent influence of past system–environment correlations.

From a physical standpoint, fractional dynamics do not preserve entanglement by shielding the system from its environment, but by **redistributing correlations across multiple scales** within the continuum. As a result, entanglement may decay more slowly or exhibit long-lived residual correlations, particularly in systems governed by power-law spectral densities or nonlocal transport mechanisms. This perspective aligns with the view of entanglement as a dynamical resource whose robustness depends not only on coupling strength, but also on the statistical structure of environmental interactions. Fractional calculus thus provides an effective framework for describing entanglement evolution in complex quantum media, where memory and scale-free coupling fundamentally alter how quantum correlations are lost, shared, or temporarily retained.

Physical Realizability and Implications of Fractional Quantum Theories

Fractional models do not propose a modification of the fundamental postulates of physics, nor do they imply new forces or exotic particles acting in everyday life. Instead, they provide **effective descriptions** of systems in which interactions span multiple spatial or temporal scales, or where microscopic degrees of freedom have been coarse-grained into a continuum. As such, fractional dynamics typically emerge in structured, disordered, or strongly coupled environments, rather than in isolated systems. For most daily-life phenomena — where interactions are short-ranged, memory effects are weak, and characteristic scales are well defined — classical or integer-order models remain entirely sufficient.

Nevertheless, fractional theories do offer explanations for observable macroscopic behaviors that are otherwise difficult to model within standard frameworks. Fractional calculus is particularly effective in describing systems where the response at a given time depends on the accumulated history of the system, or where spatial interactions extend beyond nearest neighbors. Examples include anomalous diffusion in crowded or disordered media, long-time relaxation in complex materials, non-exponential dissipation in viscoelastic systems, and wave propagation in heterogeneous structures. A notable illustration is magnetic hysteresis, where the magnetization depends not only on the instantaneous applied field but also on the entire history of field variation; fractional-order models naturally capture this path dependence and reproduce experimentally observed hysteresis loops and loss behavior that are difficult to describe using classical integer-order differential equations. While such effects may not be consciously noticeable in everyday experience, they are routinely encountered in materials science, condensed matter physics, geophysics, and photonics. In this sense, fractional calculus acts as a unifying mathematical language that captures how memory, nonlocality, and scale-free interactions shape real physical processes across multiple length and time scales.

b. Unifying Perspective: Fractional Calculus across Waves, Optics, and Quantum Systems

Fractional calculus provides a common mathematical framework for describing physical systems in which locality, scale separation, or memory assumptions break down. Across classical wave propagation, optical imaging, and quantum dynamics, fractional operators arise when energy or probability amplitude spreads over multiple spatial or temporal scales simultaneously. Rather than replacing established theories, fractional models act as effective descriptions that capture emergent behavior in complex or continuum-coupled environments.

In optical systems, fractional wave equations and fractional transfer functions naturally account for scale-free point-spread function broadening and the redistribution of energy beyond the classical diffraction envelope. Super-oscillatory fields, which rely on delicate interference among band-limited components, are especially sensitive to such nonlocal propagation, highlighting the interplay between resolution enhancement and fractional dissipation. These optical manifestations serve as experimentally accessible analogues of deeper nonlocal dynamics encountered in quantum systems.

At the quantum level, the connection between Lévy statistics and fractional Schrödinger equations reveals that nonlocal operators encode the statistical structure of underlying quantum paths. When quantum systems interact with dense continua of environmental modes, memory effects and scale-free transport become unavoidable, leading to fractional time evolution, non-exponential decoherence, and modified entanglement dynamics. Fractional calculus thus unifies descriptions of anomalous diffusion, dissipation, and coherence loss under a single mathematical language rooted in nonlocality and history dependence.

Viewed collectively, these examples demonstrate that fractional dynamics are not confined to a single physical domain, but emerge wherever complex interactions span multiple scales.

From optical imaging to quantum continua, fractional calculus offers a bridge between microscopic mechanisms and macroscopic observations, enabling a consistent interpretation of nonlocal transport, long-time memory, and scale-free energy redistribution across diverse physical systems.

Against this backdrop, it is natural to ask whether fractional dynamics and continuum-induced memory effects merely describe complex environments, or whether they can be deliberately harnessed to influence quantum information processing and computation

V. FRACTIONAL DYNAMICS AND QUANTUM COMPUTATION

Quantum computation relies fundamentally on coherence, entanglement, and controlled unitary evolution within a well-defined Hilbert space. In idealized models, quantum gates are assumed to act instantaneously and independently of environmental memory, while decoherence is treated as an external perturbation that must be suppressed. In realistic quantum hardware, however, qubits are inevitably embedded in structured environments—solid-state lattices, electromagnetic continua, or many-body reservoirs—where interactions are neither strictly local nor memoryless. Fractional dynamics provide an effective framework for describing such non-ideal conditions, capturing how nonlocal coupling and long-time correlations influence the evolution of quantum information.

From this perspective, fractional calculus does not introduce new computational primitives, but rather modifies the *dynamical background* against which quantum computation takes place. Fractional time evolution can lead to non-exponential decoherence and history-dependent noise, altering gate fidelities and error accumulation in ways not captured by Markovian models. Similarly, fractional spatial operators reflect long-range couplings and anomalous transport of quantum amplitudes, which may impact entanglement distribution and information flow in extended or networked quantum architectures. Understanding these effects is essential for accurately modeling quantum devices operating in complex environments, particularly in solid-state, photonic, and hybrid quantum platforms.

While environmental memory is traditionally viewed as detrimental to quantum computation, fractional dynamics suggest a more nuanced picture. Long-time correlations encoded through fractional operators can, under certain conditions, slow decoherence and preserve partial coherence over extended time scales. This opens the possibility that non-Markovianity and memory—when properly characterized—may be harnessed to stabilize quantum states or engineer noise-resilient operations. At the same time, the same memory effects can introduce history-dependent errors and nontrivial correlations between successive gate operations, posing challenges for conventional error-correction schemes.

1. Quantum Gates: A Geometric View

The fundamental unit of quantum computation is the qubit, whose pure states can be represented geometrically on the **Bloch sphere**. Any normalized qubit state

$$|\,\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle$$

corresponds to a point on the surface of a unit sphere, parameterized by the polar and azimuthal angles $(\theta,\, \phi)$. In this representation, quantum gates correspond to **rotations of the Bloch vector**, transforming one point on the sphere into another without changing its magnitude.

Standard single-qubit quantum gates—such as the Pauli gates, the Hadamard gate, and phase gates— are generated by integer-order Hermitian operators, typically the Pauli matrices. These gates perform well-defined rotations about fixed axes on the Bloch sphere. For example, the Pauli-X gate corresponds to a rotation by π about the x-axis, while a phase gate implements a rotation about the z-axis. In this geometric picture, quantum computation proceeds as a sequence of discrete, controlled rotations that steer the qubit state along well-defined paths on the sphere.

More generally, any unitary gate can be written as

$$U(\theta, \hat{n}) = exp\left[-i\left(\frac{\theta}{2}\right)\hat{n}\cdot\boldsymbol{\sigma}\right],$$

where $\hat{n}$ is a unit vector defining the rotation axis and θ the rotation angle. Here $\boldsymbol{\sigma}$ [or $(\sigma_X, \sigma_Y, \sigma_Z)$] denote the Pauli spin matrices, which generate rotations on the Bloch sphere and form the algebraic backbone of both standard and fractional quantum gates. This formulation emphasizes that conventional quantum gates correspond to **smooth, local rotations** generated by standard Hamiltonians over a specified interaction time.

2. Fractional Gates as Generalized Rotations

Fractional quantum gates arise when the generators of these rotations are modified to include **fractional or nonlocal operators**, reflecting underlying dynamics with memory or scale-free structure. Rather than altering the logic of quantum computation, fractional gates modify **how the Bloch vector moves on the sphere**.

In contrast to standard gates, where the trajectory between two points is determined by local, integer-order evolution, fractional gates correspond to evolution under Hamiltonians whose spectral properties follow power-law rather than quadratic dispersion. Geometrically, this means that the qubit's path on the Bloch sphere need not correspond to a simple great-circle rotation. Instead, the evolution can exhibit **non-uniform angular progression**, history dependence, or effective long-range coupling between states.

From this perspective, a fractional gate can be viewed as implementing a **generalized rotation**, one that interpolates continuously between identity and conventional unitary operations, but whose generator encodes nonlocality in time or space. The final state still lies on the Bloch sphere—as required by unitarity—but the route taken to reach it reflects the influence of fractional dynamics.

3. Physical Meaning of Fractional Gates

The appearance of fractional gates does not imply new logical operations beyond standard quantum mechanics. Rather, they emerge naturally when qubits are embedded in **structured environments** or continua, where evolution is influenced by long-time correlations or anomalous transport. In such settings, the effective Hamiltonian governing gate operations may involve fractional powers of momentum, energy, or interaction operators.

As a result, fractional gates are best interpreted as **effective gates**, capturing how real physical systems implement rotations in the presence of memory and nonlocal coupling. They provide a useful language for describing gate operations in photonic lattices, long-range spin systems, and open quantum platforms where idealized, Markovian control is not strictly applicable.

Here are the key-differences between the two types of gates

Standard gate	**Fractional gate**
Instantaneous	Time-smeared
Unitary	Generally non-unitary
Markovian	Non-Markovian
Local in time	Memory-encoded

Therefore, Fractional quantum computation lies between analog and digital quantum computing, where gates are trajectories rather than operators.

Why Fractional Quantum Operations do not Change the Hilbert Space

Fractional quantum operations modify **how a state evolves in time**, not **what the state is allowed to be**. The Hilbert space of a quantum system is fixed by its physical degrees of freedom — for example, a qubit always lives in a two-dimensional complex vector space spanned by $|0\rangle$ and $|1\rangle$, and an optical field lives in a space of square-integrable mode functions. Fractional calculus does not introduce new basis states, additional dimensions, or extended degrees of freedom; instead, it alters the temporal response with which existing operators act on those states.

This distinction is crucial. In standard quantum mechanics, unitary evolution corresponds to rotations within the same Hilbert space. Fractional evolution replaces these rotations with **history-dependent trajectories**, but the trajectories are still entirely confined to the original space. On the Bloch sphere, this is seen as spirals rather than circles: the state moves differently, but never leaves the sphere. The inward motion reflects loss of coherence, not leakage into a larger or different state space.

Memory Changes Dynamics, not State Space

Fractional dynamics encode memory through nonlocal time evolution, meaning the current state depends on its past evolution. However, this memory resides in the **evolution operator**, not in the state vector itself. At every instant, the system is still described by a valid quantum state in the same Hilbert space. No additional degrees of freedom are introduced explicitly, unlike in open-system approaches where an environment enlarges the total Hilbert space and the system is obtained by tracing out hidden variables.

This is why fractional quantum gates can model decoherence-like effects without invoking an external bath. The apparent non-unitarity emerges from the fractional temporal kernel, not from coupling to new states. As a result, fractional operations remain **local in Hilbert space**, even though they are nonlocal in time.

4. Standard Quantum gates and their Fractional Counterparts

It would be beneficial to compare the key standard quantum gates and their fractional counterparts one by one.

Physically, a quantum gate $\boldsymbol{U}$ is often generated by a Hamiltonian $\boldsymbol{H}$ acting over a time interval t

$$\boldsymbol{U} = exp\left(-\frac{i}{\hbar}\boldsymbol{H}t\right)$$

where $\boldsymbol{H}$ is a Hermitian operator ($\boldsymbol{H} = \boldsymbol{H}^{\dagger}$) representing the energy of the system. Common rotation gates (like , $\boldsymbol{R_y}$, $\boldsymbol{R_z}$) are specific cases of this where H is one of the Pauli matrices. The term $\boldsymbol{H}t/\hbar$ is dimensionless and it often appears as the phase in a wave function. The term $exp\,(-\frac{i}{h}\text{Ht})$ is the unitary evolution operator.

For unit time or its multiple this generalized quantum gate can be expressed as

$$\boldsymbol{U} = e^{-i\theta\boldsymbol{H}}$$

Confirming their instantaneous application, $U^{\dagger}U = I$, no memory (state depends only on input) and point-to-point mapping on Bloch sphere.

On the other hand, a fractional gate is defined as

$$(i\hbar)^{\alpha}\partial_t^{\alpha}\psi(t) = H\psi(t)$$

$$0 < \alpha \leq 1$$

Key object:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}$$

This immediately introduces time-smearing, non-exponential evolution, memory kernel, non-unitary contraction.

Relation between ω and Ω (Rabi frequency)

Here ω denotes the bare transition (or drive) frequency appearing in the Hamiltonian, while Ω represents the corresponding Rabi frequency governing population oscillations; in fractional quantum dynamics, this Rabi frequency becomes effectively time-dependent due to memory effects.

Small ω: Physical Transition / Drive Frequency

In our single-qubit Hamiltonian we typically write

$$H = \frac{\hbar\omega}{2}\sigma_i$$

where:

- ω is a **physical angular frequency**
- it may represent
  - level splitting (ω0), or
  - an applied field frequency, or
  - an effective coupling strength

By itself, ω is **not yet** the Rabi frequency.

Capital Ω: effective Rabi frequency

The **Rabi frequency** Ω is defined as the **rate at which population oscillates** under the action of the Hamiltonian.

For the above Hamiltonian:

$$\Omega \equiv \omega$$

(up to conventional factors of 2, depending on how the Hamiltonian is written).

Many authors prefer:

$$H = \frac{\hbar\Omega}{2}\sigma_i$$

so that:

$$\boldsymbol{U(t) = \exp\left(-i\frac{\Omega t}{2}\sigma_i\right)}$$

For **fractional time evolution**:

$$(i\hbar)^\alpha \partial_t^\alpha \psi = H\psi$$

The argument of the Mittag–Leffler function must be **dimensionless**, so we define:

$$\Omega_\alpha = \frac{\omega t^{1-\alpha}}{\hbar^{1-\alpha}}$$

Then:

$$\boldsymbol{U^\alpha(t) = E_\alpha(-i^\alpha \Omega_\alpha t^\alpha \sigma_i) = E_\alpha(-i^\alpha \omega t^\alpha \sigma_i)}$$

So:
- ω: bare frequency / coupling
- Ω: observable oscillation rate
- In fractional systems, the Rabi frequency becomes time-dependent

Bloch-Sphere Consequence
- **Standard case**:

Bloch vector rotates with constant angular velocity Ω

- **Fractional Case**:

$$\theta(t) \sim \Omega t^{\,\alpha}$$

- Rotation slows down
- Spiral trajectory
- Incomplete population transfer

Therefore, in fractional quantum dynamics, the Rabi frequency is no longer a constant but acquires a time-dependent renormalization due to memory effects.

a. Pauli-X gate (bit flip)

i. Standard Pauli-X

The analytical expressions for the $\boldsymbol{X}$ gate operator

$$\boldsymbol{X} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} = \mathbf{exp}\left(-\boldsymbol{i}\frac{\boldsymbol{\pi}}{\mathbf{2}}\boldsymbol{\sigma}_{\boldsymbol{x}}\right)$$

This shows Bloch actions

- Rotation by π about **x-axis**
- Pure rigid rotation
- Bloch vector length preserved

For a **Pauli-X gate**, the Hamiltonian describes the physical energy required to rotate the qubit's state around the X-axis of the Bloch sphere.

If we assume the gate is applied over a time interval t such that the total "pulse area" results in a π rotation (the 180 degree flip), the Hamiltonian is:

$$H = \omega X = \widehat{\omega \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}}$$

Where:

- X is the Pauli-X operator.
- ω represents the coupling strength or the frequency of the driving field (like a microwave pulse in superconducting qubits).

If we set our units such that $\hbar = 1$ and we apply the Hamiltonian for a specific time Tsuch that $\omega T = \pi/2$, the evolution becomes:

$$U(T) = \exp\{-i(\pi/2)X\} = \cos(\pi/2)\, I - i\sin(\pi/2)\, X = -iX$$

Except for a global phase of $-i$ (which doesn't change the measurement outcome), this is exactly the **Pauli-X gate**.

ii. Fractional Pauli-X

Generator

$$H_X = \frac{\hbar\omega}{2}\sigma_x$$

Fractional evolution

$$U_X{}^{(\alpha)}(t) = E_\alpha\left(-i^\alpha \frac{\omega}{2} t^\alpha \sigma_x\right)$$

Using eigenvalues ±1

$$U_X{}^{(\alpha)} = \begin{pmatrix} E_\alpha(-i^\alpha \Omega_X) & 0 \\ 0 & E_\alpha(i^\alpha \Omega_X) \end{pmatrix}$$

iii. Confirming the Four Differences

Time-Smearing Confirmed

Here the time evolution of the wavefunction can be represented as

$$|\psi(t)\rangle = \int_{t=0}^{t} K_\alpha(t-\tau)\sigma_X|\psi(\tau)\rangle d\tau$$

Whereas, for standard case $|\psi'\rangle = X|\psi\rangle$, i.e. instantaneous – not time dependent.

Not Unitary - Amplitude Contraction Happens

Because $(i^\alpha \Omega_X) < 1$ for $(\alpha < 1)$

Non-Markovian - Explicit Memory Kernel

The Fractional time derivative shows the memory effect

$$\partial_t^\alpha \psi(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t=0}^{t} \frac{\dot{\psi}(\tau)}{(t-\tau)^\alpha} d\tau$$

On the other hand, a Markovian process can model the sudden collapse of a wave function. The derivative $\frac{d\psi}{d\psi}$is mostly deterministic, broken by abrupt, random "jumps" (non-continuous changes).

Nonlocal in Time

Evolution depends on **entire past trajectory**, not just current state.

iv. Bloch Sphere Picture

Standard X	**Fractional X**
Rigid π-rotation	Spiral rotation
Fixed radius	Shrinking radius
Great circle	Contracting helix

Key Insight

Fractional X flips the qubit *while degrading coherence.*

b. Pauli-Y gate

Standard Operator Expression

$$Y = \exp(-i\frac{\pi}{2}\sigma_Y)$$

Bloch: rotation about **y-axis**

Fractional Gate Output

$$\boldsymbol{U_Y^{(\alpha)}(t) = E_\alpha(-i^\alpha \Omega \sigma_Y)}$$

Bloch Effect

- Rotation + damping
- Asymmetry in phase quadratures
- Spiral tilted toward equator

It confirms all four differences exactly as X

c. Pauli-Z Gate (phase flip)

Standard Gate Operator Expression

$$Z = \exp(-i\frac{\pi}{2}\sigma_Z)$$

Bloch:

- Rotation around **z-axis**
- No population change

Fractional Z Yields

$$\boldsymbol{U_Z^{(\alpha)}(t) = E_\alpha(-i^\alpha \Omega \sigma_Z)}$$

Pure Dephasing Emerges Naturally.

Bloch Picture

Standard Z	**Fractional Z**
Horizontal circle	Inward spiral
No decoherence	Dephasing
Fixed latitude	Radial collapse

Fundamental Difference between Fractional Z and Fractional X / Y Gates

Fractional Z gates induce pure dephasing, whereas fractional X and Y gates induce both dephasing and population relaxation.

In a physical system, "noise" usually aligns with a specific axis (typically the Z-axis for energy eigenstates). When we apply fractional gates, we are essentially rotating the state vector by an angle θ.

Fractional Z Gates: Pure Dephasing

A fractional Z gate—represented as Z^θ or (θ)—performs a rotation around the Z-axis.

- **The Math:** The operation is $\exp\left(-i\,\frac{\theta}{2}Z\right)$. If the qubit is in a superposition state $|\psi\rangle = a|0\rangle + b|1\rangle$, this gate changes the **relative phase** between $|0\rangle$ and $|1\rangle$ but does not change the magnitudes $\sqrt{a^2 + b^2}$.
- **The Result:** Since the "population" (the probability of being in $|0\rangle$ vs $|1\rangle$ stays exactly the same, there is no **population relaxation**. It only affects the phase coherence.

Fractional X and Y Gates: Mixed Effects

Fractional X (R_X) or Y (R_Y) gates rotate the state vector around the horizontal axes of the Bloch sphere.

- **The Math:** These operations are $\exp\left(-i\frac{\theta}{2}X\right)$ and $\exp\left(-i\frac{\theta}{2}Y\right)$
- **Population Relaxation:** Unlike the Z gate, these rotations move the state "up and down" between the $|0\rangle$ and $|1\rangle$ poles. If you start at $|0\rangle$ and apply a fractional X gate, you change the probability of finding the qubit in the excited state. This movement across energy levels is the definition of population relaxation.
- **Dephasing:** Because these gates move the state away from the poles and into the equatorial plane (or change its position relative to the Z-axis), they also expose the state to phase noise.
- **The Interaction:** In a real-world lab, the Z-axis is defined by the energy gap of the qubit. Any rotation that isn't perfectly aligned with that energy axis (X or Y) necessarily mixes the energy levels and the phases simultaneously.

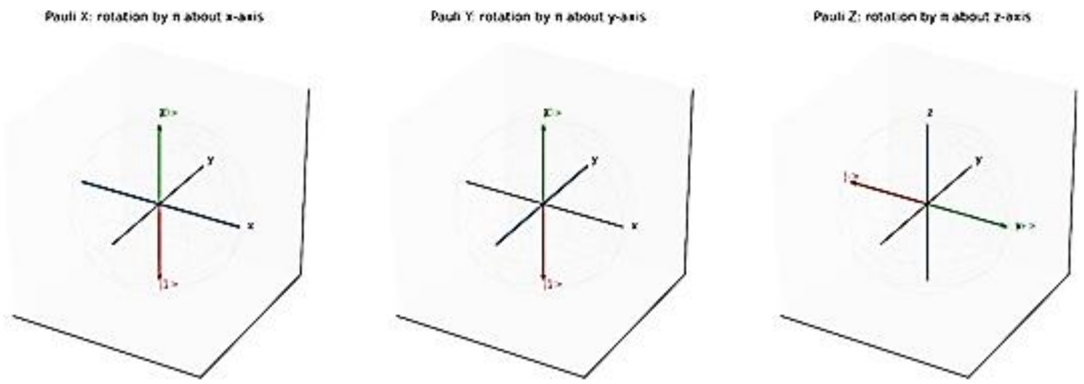

Figure 4: The Bloch Sphere with Axes X, Y, Z Is Shown. The Highlighted Rotation Axis is Blue. the Example State Vector is Drawn in Green Before Operation and Red After Operation of the Gate. Pauli

X and Y are 180° rotations that flip $|0\rangle$ ($|1\rangle$ (south pole) but around different axes; Pauli Z is a 180° rotation around the z-axis, which changes the *phase* (so it keeps $|0\rangle$ where it is but flips $|+\rangle$ to $|-\rangle$ on the equator).

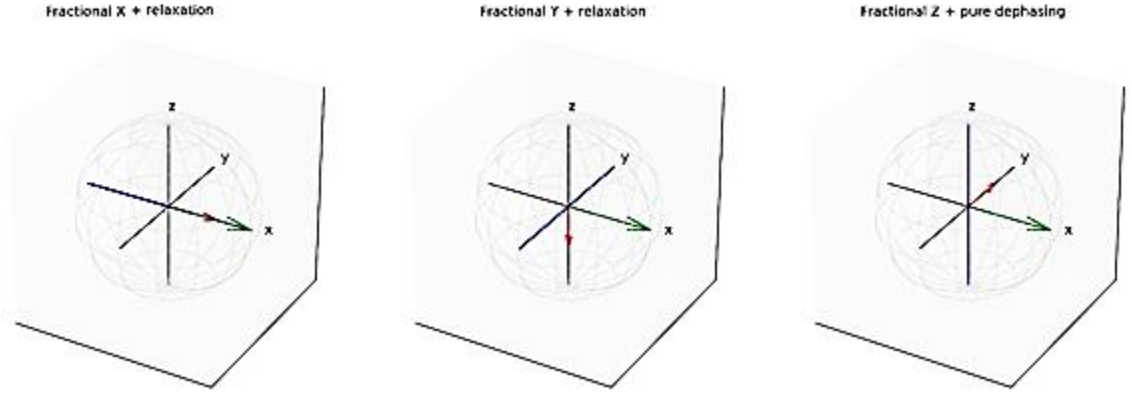

Figure 5: Fractional Pauli X, Y, Z with Decoherence. Rotation Axis Is Blue, Initial and Final State Vectors are Drawn in Green and Red Respectively. For Fractional Pauli-X Gate: the Bloch Vector Tilts Due To Fractional Rotation, Loses Length In The Y-Z Plane And Moves Toward the Ground State (Relaxation). Fractional Pauli-Y Gate: the Ame Idea Holds but Rotation Around Y-Axis Dephasing Shrinks The Component Perpendicular to that Axis and the Relaxation Pulls the Vector Toward Z=−1 . Fractional Pauli-Z Gate: Pure Dephasing Shrinks The Equatorial Components Only.

d. Hadamard gate

Standard Operator

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Equivalent Generator

$$\mathbf{H} = \mathbf{exp}\left(-\boldsymbol{i}\frac{\boldsymbol{\pi}}{\mathbf{2}}\frac{\boldsymbol{\sigma_x} + \boldsymbol{\sigma_z}}{\sqrt{\mathbf{2}}}\right)$$

Fractional Hadamard Expression with Mittag-Leffler relation

$$\boldsymbol{U_H}^{(\boldsymbol{\alpha})}(\boldsymbol{t}) = \boldsymbol{E_\alpha}\left(-\boldsymbol{i}^{\boldsymbol{\alpha}}\boldsymbol{\Omega}\frac{\boldsymbol{\sigma_x} + \boldsymbol{\sigma_z}}{\sqrt{\mathbf{2}}}\right)$$

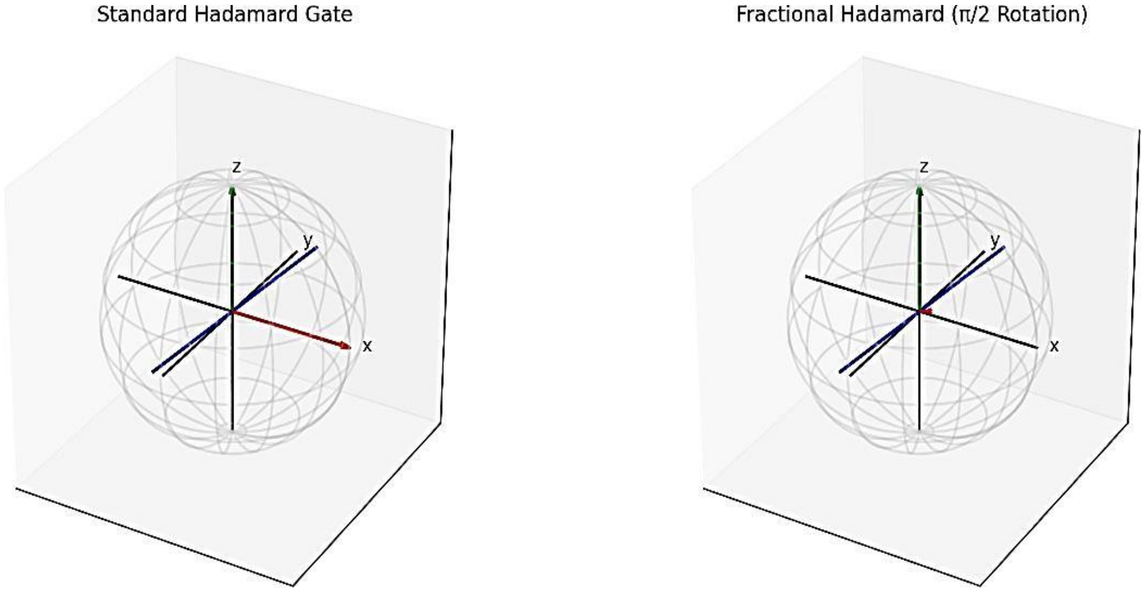

Figure 6: Green Vector is the Initial State |0⟩, Red Vector is the Final State after Applying the Gate and Blue Line is Rotation Axis of the (Fractional) Hadamard. this Pair Cleanly Illustrates How the Fractional Hadamard Moves the Bloch Vector Only Partway between |0⟩ and |+⟩, While the Full Hadamard Performs the Complete Transformation.

Bloch Consequence

- No exact great-circle mapping
- Superposition amplitudes become history-weighted
- Bloch vector contracts toward axis

Fractional H **cannot produce perfect equal superposition**

e. Phase Gate S and T

Standard

$$S = \exp(-i\frac{\pi}{4}\sigma_Z) \quad \text{and} \quad T = \exp(-i\frac{\pi}{8}\sigma_Z)$$

Fractional

$$U_{S,T}{}^{(\alpha)}(t) = E_\alpha(-i^\alpha \theta \sigma_Z)$$

- Phase accumulation becomes **nonlinear in time**
- Phase noise intrinsic

This links directly to:

- Optical dispersion
- Fractional PSF broadening
- Super-oscillatory phase fronts

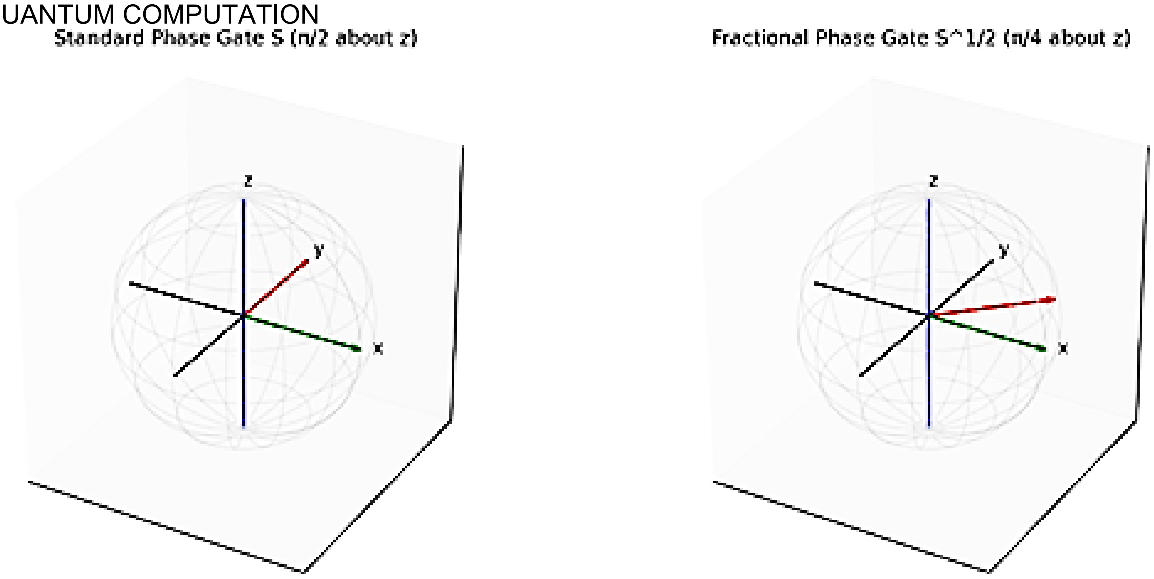

Figure 7: Green Vector is the Initial State |+⟩, Red Vector is the Final State after Applying the Gate and Blue Line Is Rotation Axis Z of the (Fractional) Phase Gate S. This Cleanly Shows how the Fractional Gate Performs *Half* the Rotation of the Standard S-Gate.

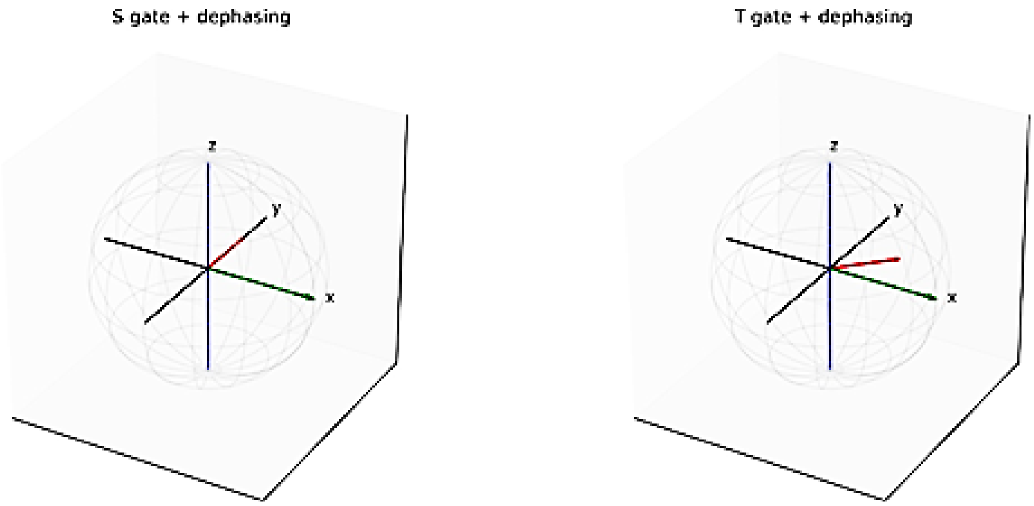

Figure 8: Green vector is the initial state |+⟩, Red vector is the final state after applying the gate and Blue line is rotation axis Z of the Phase gate S and T. Dephasing makes both the S and T final states shrink toward the z-axis. The S-gate state lands at +y (after shrinking). The T-gate state lands halfway between x and y (after shrinking).

f. CNOT Gate (two-qubit)

Standard

$$\mathrm{CNOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

Instantaneous, local, unitary.

Fractional CNOT

Generator:

$$\mathrm{H}_{CNOT} = |1\rangle\langle 1| \otimes \sigma_X$$

Fractional evolution:

$$\boldsymbol{U_{CNOT}}^{(\alpha)}(\boldsymbol{t}) = \boldsymbol{E_\alpha}(-\boldsymbol{i}^{\alpha}\mathbf{H}_{\boldsymbol{CNOT}}\boldsymbol{t}^{\alpha})$$

Consequences

- Entanglement builds **gradually**
- History-dependent correlations
- Partial entanglement saturation

Bloch (two-qubit)

- Concurrence grows sub-linearly
- No sharp entanglement threshold

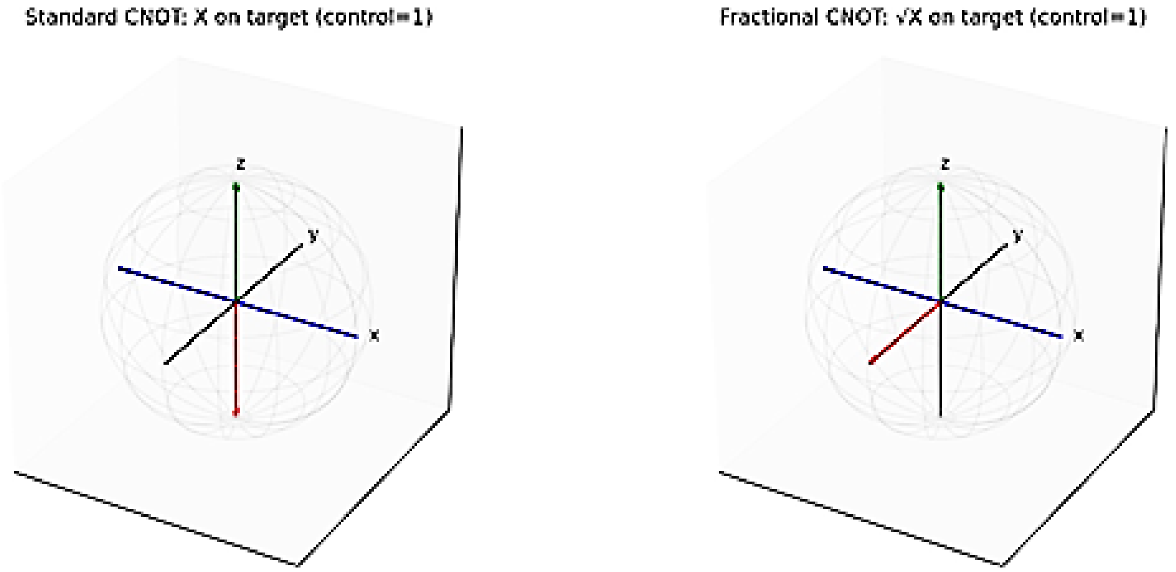

Figure 9: Green Arrow is for Initial State $|0\rangle$, Red Arrow is for Final State and the Blue Arrow is for Rotation Axis X. CNOT Rotates the Target Vector From the North Pole Straight to the South Pole. Fractional (Half) CNOT Rotates the Target Only Halfway, Producing a State on the Equator in the +Z→−Z Path.

g. SWAP Gate

Standard

$$SWAP = \exp\left[-i\frac{\pi}{4}(\sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z)\right]$$

Fractional:

$$U_{SWAP}{}^{(\alpha)}(t) = E_\alpha(-i^{\alpha}\mathrm{H}_{SWAP}\boldsymbol{t}^{\alpha})$$

- Incomplete exchange
- Memory-induced asymmetry
- Information leakage to environment

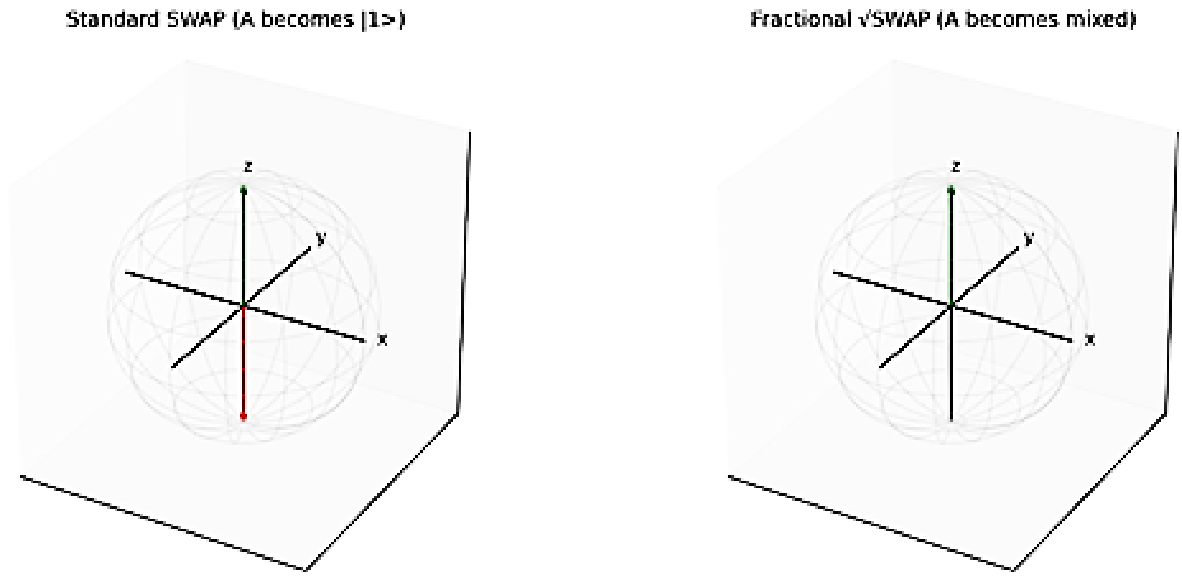

Figure 10: The Green Vector is the Initial |0⟩ State, the Red Vector is the Resulting Reduced State of Qubit a. Standard SWAP Moves Qubit a Clean Showing Deterministic Population Exchange, While the Fractional √SWAP Sends Qumixed State (Center), Because √SWAP Angles a that Shows Partial Exchange + Entanglement.

Finally we can write Fractional quantum gates are not logical primitives — they are physical processes. Here is a comparative summary of the properties of the standard and fractional quantum gates.

Property	Standard gates	Fractional gates
Time	Instantaneous	Time-smeared
Unitarity	Preserved	Broken
Memory	None	Built-in
Locality	Point operation	History-dependent
Bloch motion	Rotation	Spiral contraction

5. Fractional Quantum Gates as Optical Propagation Operators

In quantum optics and wave optics, free propagation itself can be interpreted as a continuous gate operation acting on the modal state of the field. In the standard case, propagation over time or distance generates a unitary evolution of the form

$$U(t) = \exp(-i\Omega t\hat{G})$$

where G is the generator of evolution, such as a Pauli operator for a two-level system or a quadratic phase operator for spatial modes. This evolution produces rigid rotations on the Bloch sphere for qubits and preserves full phase coherence in optical fields. Consequently, the point-spread function (PSF) retains its ideal diffraction-limited form, since all spatial frequencies interfere with well-defined relative phases.

In contrast, fractional quantum gate evolution replaces the exponential propagator by a Mittag–Leffler response,

$$U_\alpha(t) = E_\alpha(-i^\alpha \Omega t^\alpha \hat{G}) \ , \ 0 < \alpha < 1,$$

introducing intrinsic memory into the dynamics. The system no longer evolves instantaneously or Markovianly; instead, its present state depends on its entire evolution history. On the Bloch sphere, this manifests as inward spirals rather than closed circular trajectories. Importantly, this loss of coherence is not due to stochastic noise or coupling to an external bath, but arises deterministically from the fractional order of the evolution operator itself.

6. Fractional Z Gates and Optical Phase Decoherence

The fractional Z gate plays a particularly important role in optical systems because its generator is diagonal. For a two-level system, fractional Z evolution preserves populations while progressively suppressing phase coherence, as captured by

$$U_Z^\alpha(t) = E_\alpha(-i^\alpha \Omega t^\alpha \sigma_Z)$$

On the Bloch sphere, this corresponds to spiraling trajectories confined to the equatorial plane: the longitudinal component remains fixed while transverse components decay. Qualitatively, this behavior is identical to pure dephasing, but with a crucial distinction — the decay follows a Mittag–Leffler law rather than an exponential one, reflecting long-memory dynamics.

In an optical context, the same mathematics describes phase evolution across a pupil or aperture. A complex field

$$E(r) = A(r)\exp(i\phi(r))$$

subjected to fractional Z-type evolution acquires phase correlations that decay algebraically rather than abruptly. While the local amplitude (r) remains intact, long-range phase correlations across the aperture are gradually suppressed. This is optical decoherence: phase information is not destroyed instantaneously, but becomes increasingly nonlocal and history-dependent.

7. From Fractional Phase Decoherence to PSF Broadening

The point-spread function of an imaging system is determined by the coherent superposition of all pupil contributions,

$$PSF(\rho) = |\mathcal{F}\{P(r)\exp(i\phi(r))\}|^2$$

where $P(r)$ is the pupil function. In the fully coherent ($\alpha = 1$) case, phase relations are rigid, and constructive interference produces a sharp central lobe with well-defined sidelobes. Under fractional phase evolution, however, the phase factor $\exp(i\phi(r))$ carries memory-dependent correlations: distant points in the pupil no longer interfere perfectly, even though no random noise has been introduced.

As a result, the Fourier summation that forms the PSF undergoes partial dephasing. The central peak broadens, sidelobes are smeared, and the effective resolution is reduced. This phenomenon can be interpreted as **fractional PSF broadening**, directly analogous to the inward spiral observed for fractional Z gates on the Bloch sphere. The geometry of the spiral translates into incomplete interference in the spatial domain.

It is to be noted that, the Rayleigh criterion remains fundamentally intact, but its operational manifestation is modified: the central lobe widens and contrast between neighboring features diminishes gradually, following a Mittag–Leffler decay rather than an abrupt cutoff. In this sense, fractional optics alters how resolution degrades, not the diffraction limit itself. In other words, Fractional PSF broadening does not necessarily violate the Rayleigh limit, but rather reshapes it through memory-dependent coherence loss.

8. An Unified Interpretation: Fractional Gates as Imaging Operators

From this perspective, optical decoherence and PSF broadening can be viewed as the cumulative effect of repeated fractional Z-gate operations acting on the spatial-mode structure of the field. Each infinitesimal propagation step applies a non-unitary, memory-dependent phase operation, preserving local intensities while progressively eroding global phase coherence. The measured PSF is then the optical analogue of a quantum measurement performed after fractional gate evolution.

This unified framework highlights a deep connection between fractional quantum gates, non-Markovian dynamics, and imaging theory. What appears as reduced resolution in optics is, in fact, the geometric signature of fractional phase evolution — the same mechanism that converts circular Bloch trajectories into spirals. In this sense, fractional calculus provides a common language for describing decoherence in quantum systems and resolution loss in optical imaging, without invoking external noise or stochastic averaging.

9. Local Fractional Hamiltonians and Non-Unitary Evolution

Fractional quantum gates arise from modifying the **time evolution generated by a local Hamiltonian**, not from altering the Hamiltonian itself. Consider a standard single-qubit Hamiltonian

$$H = \hbar\Omega\sigma_i$$

which generates unitary rotations through exp). $\left(\frac{iHt}{h}\right)$ In fractional dynamics, the generator remains the same local Hamiltonian, but the time evolution is governed by a fractional time operator. The resulting propagator takes the form

$$\boldsymbol{U^{\alpha}(t) = E_{\alpha}(-i^{\alpha}\Omega_{\alpha}t^{\alpha}\sigma_i)}$$

where the Mittag–Leffler function encodes memory and dissipation. Crucially, the Hamiltonian still acts **locally within the same Hilbert space**; only the temporal response to that Hamiltonian is altered.

10. Energy Dissipation without Hilbert-Space Leakage

The non-unitarity of fractional gates reflects **energy dissipation or redistribution over time**, not energy transfer to explicitly modeled external degrees of freedom. Fractional evolution spreads the system's response over its history, effectively slowing down coherent energy exchange. As a result, oscillations driven by the Hamiltonian decay algebraically rather than persisting indefinitely. On the Bloch sphere, this appears as inward spiraling trajectories: the direction of motion is still determined by the Hamiltonian axis, but the radius shrinks due to dissipative memory effects.

Importantly, this dissipation does not imply that the state leaves the Hilbert space. No new energy levels are introduced, and no auxiliary modes are coupled in explicitly. The system remains fully described by the same state vector or operator at all times. The apparent loss of energy is encoded in the **nonlocality in time**, not in a spatial or spectral leakage into additional degrees of freedom.

11. Locality Preserved Despite Non-Unitarity

Locality in Hilbert space is preserved because the Hamiltonian continues to act on the same subsystem and basis states. Fractional gates do not introduce couplings between distinct qubits or modes; they only modify how the system responds to its own Hamiltonian over time. In this sense, fractional gates are local dissipative operations: they affect coherence and energy within the subsystem but do not redistribute amplitudes across subsystems. The non-unitarity arises because the evolution is no longer generated by a one-parameter unitary group, but it remains a well-defined single-system process.

This distinction explains why fractional quantum gates can be both non-unitary and local. The gate acts locally in Hilbert space while being temporally extended and dissipative. The evolution is therefore irreversible in time but not delocalized in space or state structure.

12. Optical Analogy

In optics, fractional phase evolution provides a clear analogy. Applying a fractional phase operator to a wavefront alters the coherence of the field without coupling it to another beam or optical path. The operation is local to the field, although interference patterns degrade over time or distance even in vacuum. Similarly, fractional quantum gates act locally on qubits or modes, while their non-unitarity reflects accumulated memory rather than spatial delocalization.

Physically, fractional quantum evolution can be viewed as a system evolving under a local Hamiltonian while interacting implicitly with a structured background that stores memory but does not appear as an explicit environment. Energy is not conserved instantaneously but is redistributed over long timescales, producing dissipation without Hilbert-space enlargement. This is why fractional gates simultaneously encode energy loss, non-unitarity, and locality.

Fractional quantum gates preserve Hilbert-space locality because they are generated by local Hamiltonians, while their non-unitarity reflects energy dissipation arising from temporal memory rather than coupling to additional degrees of freedom.

VI. ENTANGLEMENT, COMMON WAVEFUNCTIONS, AND LOCAL HAMILTONIANS

Entanglement arises when a composite quantum system is described by a **single, inseparable wavefunction** in the joint Hilbert space, rather than as a product of subsystem states. For two qubits A and B, this means the state cannot be written as $|\psi_A\rangle \otimes |\psi_B\rangle$, but instead occupies the full tensor-product space $H_A \otimes$. This structural property of the Hilbert space is independent of the specific dynamical law governing time evolution. In standard quantum mechanics, entanglement is generated by local interaction Hamiltonians—such as bilinear couplings between Pauli operators—that act simultaneously on both subsystems while preserving the overall Hilbert-space structure.

A typical entangling Hamiltonian,

$$H_{int} = \hbar\Omega\, \sigma_i^{(A)} \otimes \sigma_j^{(B)}$$

generates correlations by coherently redistributing amplitudes within the joint wavefunction. Under conventional unitary evolution, this redistribution occurs smoothly and reversibly in time, leading to periodic entanglement generation and decay. The Hamiltonian acts locally in Hilbert space, but its action creates nonlocal correlations between subsystems—this is the standard mechanism underlying gates such as CNOT and SWAP.

1. Lévy Flights, Time Smearing, and Fractional Entanglement Dynamics

Fractional quantum dynamics modifies this picture by replacing smooth, Brownian-like time evolution with **Lévy-flight–dominated temporal behavior**. In a Lévy process, the system undergoes rare but long temporal jumps, producing heavy-tailed waiting-time distributions and long memory. When such dynamics underlie quantum evolution, the interaction Hamiltonian remains unchanged, but its action becomes **smeared over time**. The wavefunction no longer evolves through infinitesimal, evenly spaced phase increments; instead, it accumulates correlations through sporadic, history-dependent contributions.

Mathematically, this leads to fractional time evolution governed by Mittag–Leffler functions rather than exponentials, and physically it means that entanglement is built up and degraded through a sequence of temporally nonlocal interaction events. The joint wavefunction remains defined in the same tensor-product Hilbert space, but its structure reflects the Lévy-flight statistics of the underlying dynamics. Entanglement is therefore neither instantaneously generated nor abruptly destroyed; it emerges and decays with memory, mirroring the anomalous transport associated with Lévy flights. In this sense, fractional Hamiltonians encode a form of entanglement dynamics that is locally generated, temporally extended, and intrinsically non-Markovian.

For two entangled qubits, fractional dynamics does not require a modified interaction Hamiltonian. The same bilinear Hamiltonian that generates entanglement in standard quantum mechanics remains valid and local in Hilbert space. Fractional behavior enters through the temporal evolution governed by a fractional Schrödinger equation, whose solution involves a Mittag–Leffler propagator. This propagator reflects Lévy-flight–like temporal statistics, leading to entanglement generation and decay that are history-dependent and dissipative, while preserving the tensor-product structure of the joint Hilbert space.

For intuition, one may introduce an effective fractional generator for the Hamiltonian:

$$H_{eff}^{(\alpha)}(t) \sim \hbar\Omega t^{\alpha-1} \sigma_i^{(A)} \otimes \sigma_j^{(B)}$$

which reflects the **time-weighted action** of the interaction. Here we note that, this is **not** a true Hamiltonian operator, it is just a tool for expressing memory accumulation.

2. From Unitary Gates to Memory-Weighted Gates

In standard quantum computation, two-qubit gates such as **CNOT** and **SWAP** are idealized as instantaneous unitary operations. They act through sharply defined interaction intervals, during which entanglement is either created or redistributed in a reversible manner. This picture implicitly assumes Markovian dynamics, where the system has no memory of its past evolution.

Fractional quantum gates replace this instantaneous action with **memory-weighted evolution**. The logical structure of the gate remains intact, but the gate action is distributed over time according to a fractional kernel. As a result, entanglement is generated or redistributed gradually, with the present state reflecting the accumulated history of the interaction.

3. Fractional CNOT: Conditional Dynamics with Memory

The standard CNOT gate can be generated by a Hamiltonian of the form

$$H_{CNOT} = \hbar\Omega \, |1\rangle \langle 1|^{(A)} \otimes \sigma_x^{(B)}$$

which conditionally flips the target qubit based on the control qubit's state. Under unitary evolution, this interaction produces maximal entanglement in a single gate time.

In the fractional case, the Hamiltonian remains unchanged, but its action is governed by fractional time evolution:

$$U_{CNOT}^{(\alpha)}(t) = E_\alpha \left(-i^\alpha \Omega t^\alpha \, |1\rangle \langle 1|^{(A)} \otimes \sigma_x^{(B)}\right)$$

Here, the control-target correlation is built up through Lévy-flight-like temporal dynamics. The conditional operation is no longer instantaneous; instead, the target qubit "remembers" the past influence of the control qubit. Entanglement emerges progressively and decays algebraically rather than abruptly, reflecting intrinsic dissipation without environmental coupling.

4. Fractional SWAP: Gradual Redistribution of Quantum Correlations

The SWAP gate exchanges the states of two qubits and can be generated by an exchange Hamiltonian,

$$H_{SWAP} = \hbar\Omega(\sigma_x^{(A)}\sigma_x^{(B)} + \sigma_y^{(A)}\sigma_y^{(B)} + \sigma_z^{(A)}\sigma_z^{(B)})$$

In the unitary regime, this Hamiltonian produces a perfect exchange of amplitudes and preserves all correlations.

Under fractional evolution, the corresponding propagator,

$$U_{SWAP}^{(\alpha)}(t) = E_\alpha \left(-i^\alpha \Omega t^\alpha H_{SWAP}/\hbar\right)$$

leads to a **partial and irreversible redistribution** of amplitudes. Instead of a sharp exchange, correlations flow gradually between the qubits, with memory suppressing perfect reversibility. Entanglement is not destroyed but becomes progressively mixed and history-dependent.

5. Locality Preserved, Reversibility Lost

Both fractional CNOT and fractional SWAP preserve locality in Hilbert space: they act only on the intended pair of qubits and do not introduce auxiliary systems. Their generators remain Hermitian, and the energy spectrum is unchanged. What is lost is strict reversibility, a direct consequence of temporal nonlocality. This loss is not a defect but a feature, reflecting realistic conditions where entangling operations occur in structured environments, such as dispersive optical media or solid-state systems with long-time correlations.

Fractional CNOT and SWAP gates retain logical and spatial locality while encoding memory-dependent entanglement through fractional time evolution. Entanglement is generated and redistributed as a history-weighted process governed by Lévy-like dynamics, yielding non-unitary yet energetically consistent operations.

6. Entanglement Robustness and Limits of Quantum Computation with Memory

Fractional quantum gates introduce a notion of entanglement robustness that differs from both ideal unitary evolution and standard Markovian noise models. Because entanglement decays algebraically rather than exponentially, correlations can persist over extended times even in the presence of dissipation.

At the same time, this persistence comes with an important trade-off. The same memory effects that stabilize correlations also limit controllability and reversibility: gate operations cannot be sharply switched on and off, and their action depends on the prior evolution of the system. As a result, fractional dynamics impose practical constraints on quantum computation, favoring architectures in which gradual entanglement buildup and tolerance to dissipation are more relevant than fast, fully reversible logic operations.

7. Optical and Photonic Realizations of Fractional Entangling Dynamics

Optical and photonic systems provide natural platforms for fractional entangling dynamics due to their intrinsic long-range correlations. Multimode fibers, dispersive waveguides, and propagation through structured or turbulent media can induce effective Lévy-type evolution.

In these systems, entangling interactions between spatial, polarization, or frequency modes remain local in Hilbert space, while fractional phase accumulation introduces memory-dependent coherence loss. Optical propagation distance effectively plays the role of fractional time, making photonics an accessible testbed for memory-weighted quantum logic.

8. Fractional Dynamics, Entanglement Memory, and Quantum Communication

Quantum communication and quantum key distribution (QKD) rely on the reliable distribution of quantum correlations over space and time. In idealized descriptions, these processes are often treated as sharply defined and memoryless. In realistic channels—such as optical fibers, free-space links, and integrated photonic networks—this assumption breaks down. Dispersion, turbulence, and material inhomogeneities introduce temporal smearing that aligns naturally with fractional dynamical descriptions.

Under such conditions, entanglement between distributed systems builds up gradually rather than being established instantaneously. Likewise, it decays progressively rather than vanishing abruptly, retaining partial correlations over extended times. This can enhance robustness against localized disturbances, but it also implies that successive transmissions become statistically correlated due to channel memory.

9. Implications for QKD Security and Performance

These features have important implications for QKD. On one hand, the gradual decay of entanglement can soften the impact of noise, allowing useful correlations to persist even when instantaneous fidelity is reduced—an advantage in long-distance or high-loss scenarios.

On the other hand, channel memory challenges a key assumption underlying many security analyses: that successive quantum signals are independent. Temporal correlations introduced by fractional dynamics may, in principle, be exploited by an eavesdropper if not properly accounted for. Security must therefore be understood as a history-dependent property, not solely determined by instantaneous error rates.

From this perspective, fractional quantum dynamics provide a physically grounded explanation for why real communication channels exhibit neither ideal coherence nor simple exponential decoherence. The same memory effects that govern fractional gate dynamics naturally arise during transmission, suggesting that communication strategies can be adapted to exploit memory while maintaining security.

10. Fractional Channel Models and the Breakdown of Standard QKD Assumptions

Most standard QKD security proofs rely on the assumption that transmitted quantum states are independent and identically distributed, allowing each signal to be analyzed separately.

Fractional channel models explicitly violate this assumption by introducing correlations across successive transmissions.

In such channels, correlations decay algebraically rather than exponentially, so errors and phase fluctuations are no longer statistically independent. As a result, security estimates based on finite samples may underestimate information leakage, particularly if correlations accumulate over time.

11. Memory as a Potential Eavesdropping Resource

In conventional QKD models, an eavesdropper's interaction is assumed to be local in time: each interception disturbs the quantum state independently and is detectable via increased error rates. Fractional dynamics change this picture. Because the channel itself has memory, even small, distributed interactions can coherently influence future states. An eavesdropper does not need to perform strong measurements; instead, they may exploit the channel's intrinsic memory to correlate their probes with the evolving quantum signal over extended durations.

Importantly, this does not imply a failure of quantum security, but rather a **shift in the threat model**. Security can no longer be assessed solely through instantaneous disturbance; it must account for history-dependent correlations. Fractional channel descriptions thus provide a controlled way to model realistic attacks that are invisible under Markovian assumptions but physically plausible in long-distance optical fibers, free-space links, or satellite channels.

12. Toward Memory-Aware Quantum Security

Viewed constructively, fractional channel models do not undermine QKD—they refine it. By explicitly incorporating time-smearing and memory into the channel description, one can derive security bounds that remain valid even when the i.i.d. assumption fails. This naturally motivates memory-aware countermeasures such as adaptive key distillation, temporal randomization of signal emission, or protocol designs that deliberately erase channel memory. In this sense, fractional calculus offers not just a warning about potential vulnerabilities, but a **systematic theoretical framework** for securing quantum communication in realistic, non-Markovian environments.

Outlook: Fractional QKD as a Future Research Direction

Fractional quantum channel models suggest a natural extension of quantum key distribution beyond idealized, memoryless assumptions toward frameworks that explicitly incorporate temporal correlations and channel history. As global quantum communication networks expand in scale and complexity, such non-Markovian effects will become increasingly relevant rather than negligible. Fractional QKD, viewed in this light, is not a new protocol but a refined theoretical lens through which security, robustness, and information leakage can be reassessed in realistic environments. By treating memory as a fundamental channel property—rather than an imperfection to be ignored—fractional approaches may guide the development of next-generation QKD systems that remain secure under long-range temporal correlations and imperfect channel resetting.

VII. SUMMARY

This study presents a unified framework linking fractional calculus with quantum dynamics, quantum gates, and information processing. By introducing fractional time evolution via Mittag–Leffler propagators, memory, dissipation, and temporal nonlocality are incorporated into quantum operations without modifying the Hilbert space or violating fundamental principles such as energy conservation and locality. In this setting, fractional quantum gates arise as history-dependent extensions of standard gates, preserving their logical structure while relaxing assumptions of instantaneous, unitary, and Markovian evolution.

Within this framework, single- and two-qubit operations become gradual, memory-driven processes. Fractional Pauli, phase, and entangling gates describe population transfer, dephasing, and correlation buildup as algebraic, time-distributed effects. Entanglement is correspondingly reshaped rather than destroyed, evolving over extended temporal windows with progressive decay, consistent with Lévy-like dynamics. This provides a physically grounded account of irreversibility without invoking external decoherence models or enlarged state spaces.

The approach extends naturally to multi-qubit interactions, with fractional CNOT and SWAP gates acting as memory-weighted entangling operations that preserve Hilbert-space locality while introducing controlled temporal irreversibility. Optical and photonic systems offer promising platforms for realizing such dynamics, where dispersion and structured propagation environments give rise to fractional behavior. Effects such as PSF broadening, super-oscillation, and optical decoherence admit a unified interpretation within this framework.

Finally, implications for quantum communication and security are considered. Fractional channel models introduce long-range temporal correlations that challenge standard assumptions in quantum key distribution, while also motivating memory-aware security strategies. Overall, fractional quantum dynamics emerges not merely as a mathematical extension, but as a practical framework for understanding and engineering quantum systems in complex environments.

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